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AVERAGING HERMITE INTERPOLATORS AND THEIR CONVERGENCE PROPERTIES

by

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ABSTRACT

The concept of averaging interpolation due to Motzkin, Sharma and Straus may be considered as a generalization of Lagrange interpolation. It is then natural to try to find an analogous generalization of Hermite interpolation. We seek to provide such a generalization and to study some of its convergence properties. Chapter 0 contains a brief survey of relevant results from the vast literature on Lagrange and Hermite interpolation, as well as a summary of recent results on averaging interpolation.

Chapter I deals with averaging Hermite interpolation both for algebraic and trigonometric polynomials. Chapters II and III are concerned with the convergence problem on different nodes. While Chapter II deals with nodes which are the zeros of $(1-x^2)T_n(x)$, Chapter III is concerned with Jacobi abscissas. We also show in Chapter II that averaging Hermite interpolators offer some advantage over Hermite-Féjer interpolators with zero derivative at the nodes. In Chapter IV we discuss the convergence of trigonometric averaging interpolators on equidistant nodes.

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CHAPTER I

PROLEGOMENA

1. Introduction.

Let $E_n = \{x_{1n},\ldots,x_{nn}\}$, $1 \geq x_{1n} > \ldots > x_{nn} \geq -1$ denote the n-th row of a triangular matrix E and let $f(x)$ be defined on $[-1,1]$. For simplicity, we shall often write x_k for x_{kn}, $k = 1,\ldots,n$. The polynomials of Lagrange and Hermite interpolation at the points of E_n, often denoted by $L_{n-1}(f,x)$ and $H_{2n-1}(f,x)$ respectively, have been studied in great detail for various choices of E for their convergence behaviour as n tends to infinity. In 1914, Faber showed ([18], Vol. III, Theorem 2, p. 27) that, for every E, there exists a function $f \in C[-1,1]$ such that $L_{n-1}(f,x)$ does not converge uniformly to $f(x)$ on $[-1,1]$. We then have the classical result of Féjer that, if E_n consists of the zeros of $T_n(x)$, the Chebyshev polynomial of first kind, then $H^{\circ}_{2n-1}(f,x)$, the Hermite-Féjer interpolator with zero derivatives at such points, converges uniformly to $f(x)$ on $[-1,1]$ whenever $f \in C[-1,1]$.

Among the many diverse results in this area, we select for later reference some recent results of Berman, Saxena and Riess.

2. Hermite-Féjer Interpolation.

Let $w(x) = (x-x_1)\ldots(x-x_n)$ and $\ell_k(x) = w(x)/(x-x_k)w'(x_k)$, $k = 1,\ldots,n$. Given a function $f(x)$ and a vector $y' = (y_1',\ldots,y_n')$, the Hermite-Féjer interpolator $H_{2n-1}(f,x) \equiv H_{2n-1}(f,w;x)$ defined by

$$(2.1)\qquad H_{2n-1}(f,x_k) = f(x_k)\ ,\quad H'_{2n-1}(f,x_k) = y'_k\ ,\quad k = 1,\dots,n$$

is given by the formula

$$(2.2)\qquad H_{2n-1}(f,x) = \sum_{k=1}^{n} f(x_k)h_k(x) + \sum_{k=1}^{n} y'_k h^*_k(x)\quad ,$$

where for the fundamental polynomials of Hermite-Féjer interpolation $h_k(x)$ and $h^*_k(x)$ we have the explicit formulae:

$$(2.3)\qquad \begin{aligned} h_k(x) &= \left(1-(x-x_k)\frac{w''(x_k)}{w'(x_k)}\right)\ell_k^2(x) = \\ &= \frac{-w''(x_k)}{(w'(x_k))^3}x^{2n-1} + \frac{1+(2s_n-x_k)w''(x_k)/w'(x_k)}{(w'(x_k))^2}x^{2n-2} + \dots \end{aligned}$$

$$(2.4)\qquad \begin{aligned} h^*_k(x) &= (x-x_k)\ell_k^2(x) = \\ &= \frac{1}{(w'(x_k))^2}x^{2n-1} - \frac{2s_n-x_k}{(w'(x_k))^2}x^{2n-2} + \dots \ . \end{aligned}$$

In the special case when $y' \equiv 0$, we shall denote (2.2) by $H^{\circ}_{2n-1}(f,x)$.

In 1965 and 1969, Berman ([1], [2]) has shown that, when $w(x) = (1-x^2)T_{n-2}(x)$, the polynomials $H^{\circ}_{2n-1}(f,w;x)$ diverge at all points of $(-1,1)$, except perhaps at $x = 0$, when $f(x) = x$, $|x|$ and x^2. Later, in 1970, he showed [3] that, if $\tilde{H}^{\circ}_{2n-2}(f,w;x)$ denotes the extended H-F polynomial of degree

2n-2 uniquely determined by

$$\tilde{H}^{\circ}_{2n-2}(f,x_k) = f(x_k), \qquad k = 1,\ldots,n-1$$

$$\tilde{H}^{\circ\prime}_{2n-2}(f,x_k) = 0 , \qquad k = 1,\ldots,n ,$$

and if $f = x, x^2$ then $\tilde{H}^{\circ}_{2n-2}(f,x)$ diverges for every point on $(-1,1)$. A similar result for $f(x) = |x|$ was obtained by Saxena [20] in 1967. These results are striking when compared with Féjer's result [8] and with a recent result of Riess. Riess [19] showed that if $w(x) = (1-x^2)U_{n-2}(x)$, where $U_n(x)$ are the Chebyshev polynomials of second kind, then $H^{\circ}_{2n-1}(f,w;x)$ converges to $f(x)$ uniformly on $[-1,1]$, for all $f \in C[-1,1]$.

3. Some Linear Polynomial Operators.

Recently the polynomials of Lagrange interpolation have been generalized in two different ways by Motzkin and Sharma [16] and Motzkin, Sharma and Straus [17]. In the first generalization, starting with the fundamental polynomials of Lagrange interpolation $\ell_k(x) \equiv \ell_{ko}(x)$, Motzkin and Sharma [16] define inductively, for a given integer $r \leq n-1$, a sequence of fundamental polynomials $\ell_{kr}(x)$, $k = 1,\ldots,n$. With these $\ell_{kr}(x)$ a linear polynomial operator $L_{nr}(f,x)$ of degree $n-r-1$ is constructed:

$$L_{nr}(f,x) = \sum_{k=1}^{n} f(x_k)\ell_{kr}(x) .$$

These two authors show that $L_{nr}(f,x)$ has many properties similar to those of $L_{no}(f,x)$, the Lagrange interpolator at the nodes E_n in the authors' notation. Among the results proved in [16] are the following:

(i) For any point matrix E there exists a function $f \in C[-1,1]$ to which $L_{nr}(f,x)$ fails to converge uniformily on $[-1,1]$. (Analogue of Faber's result for $L_{no}(f,x)$ ([18], Vol. III, Theorem 2, p. 27).)

(ii) If $x_1,\ldots,x_n$ are the zeros of an orthogonal polynomial with weight $w(x) \geq 0$ and $f \in C[-1,1]$ then

$$\lim_{n \to \infty} \int_{-1}^{1} [L_{nr}(f,x)-f(x)]^2 w(x)\,dx = 0 .$$

(Analogue of Erdős-Turán's result [7] for $L_{no}(f,x)$).

(iii) If the $x_1,\ldots,x_n$ are the zeros of $T_n(x)$ and $f \in C[-1,1]$, then

$$\lim_{n \to \infty} \int_{-1}^{1} [L_{nr}(f,x)-f(x)]^4 dx = 0 .$$

(Analogue of Feldheim's result ([9], Theorem 1, p. 78)).

4. Averaging Interpolation.

For $r = 1$, the polynomials $L_{nr}(f,x)$ of §3 are the next-to-interpolatory polynomials of degree $n-2$ [15] which are polynomials of best approximation in the uniform norm on the point set E_n. For $r > 1$ a similar approximation-theoretic property is not available.

This led Motzkin, Sharma and Straus to introduce another class of polynomials [17]. Let $A_m(z) = \sum_o^m a_k z^k$ be a given polynomial of degree $m \leq n-1$ with negative roots. Consider the $n-m-1$ linear functionals

$$(4.1) \qquad L_j(f) = \sum_{k=o}^{m} a_k f(x_{k+j}) \ , \qquad j = 1,\ldots,n-m-1 \ .$$

The authors showed that, for every $f(x)$, there exists a unique polynomial $P(x)$ of degree $\leq n-m-1$ such that

$$(4.2) \qquad L_j(P) = L_j(f) \ , \qquad j = 1,\ldots,n-m-1 \ .$$

Such polynomials $P(x)$ have the following approximation theoretic meaning. If

$$A_m(z) = (1+\alpha_1 z) \ldots (1+\alpha_m z)$$

and

$$\frac{A_m(z)}{1+\alpha_r z} = \sum_{k=o}^{m-1} a_{kr} z^k$$

then among all polynomials of degree $\leq n-m-1$, $P(x)$ minimizes

$$\max_{j=1,\ldots,n-m-1} \alpha_r^{j-1} \left| \sum_{k=o}^{m-1} a_{kr} \delta_{k+j} \right|$$

where $\delta_k = P(x_k)-f(x_k)$, $k = 1,\ldots,n$.

The polynomials $P(x)$ above will be denoted throughout by $L_\nu(A_m,f;x)$, $\nu = n-m-1$ and called averaging interpolators, since certain averages of values, rather than function values, as prescribed. Note that when $m = 1$, these polynomials reduce to the next-to-interpolatory polynomials mentioned above.

5. Convergence Behaviour of $L_\nu(A_m,f;x)$.

Saxena and Sharma [21] have obtained some properties of $L_\nu(A_m,f;x)$ when $m = 2$, $A_2(z) = 1+2z+z^2$, which are similar to those of the Lagrange interpolators $L_{n-1}(f,x)$. For example:

(i) if E_n consists of the zeros of either $T_n(x)$ or $(1-x^2)U_{n-2}(x)$ and $f(x)$ is in the Dini-Lipschitz class on $[-1,1]$, then $L_{n-3}(A_2,f;x)$ converges to $f(x)$ uniformly on $[-1,1]$;

(ii) if E_n consists of the zeros of $T_n(x)$ and $f \in C[-1,1]$, then

$$\lim_{n\to\infty} \int_{-1}^{1} \{L_{n-3}(A_2,f;x)-f(x)\}^2 \frac{dx}{\sqrt{1-x^2}} = 0 .$$

6. Table of Known Results.

For convenience, we list here some of the known convergence properties of the operators mentioned above for a variety of point matrices E. The numbers in brackets are bibliographical references.

Operator	Uniform Convergence Classes for $w = w(x) =$				
	$T_n(x)$	$P_n^{(\alpha,\beta)}(x)$	$;\alpha,\beta > -1$	$(1-x^2)T_{n-2}$	$(1-x^2)U_{n-2}$
$L_{n-1}(f,w;x)$	Dini-Lip. on [-1,1] [23]	$\alpha,\beta \leq -1/2$: Dini-Lip. on [-1,1] [23]	α or $\beta > -1/2$: Dini-Lip.$^{(*)}$ on [-1,1] [23]	[5]	[13]
$L^{*}_{n-1}(f,w;x)$	C[-1,1] [11]	$\alpha,\beta \leq -1/2$: $C[-1,1]^{(*)}$ [26]	α or $\beta > -1/2$: $C[-1,1]^{(*)}$ [26]		
$L^{\circ}_{n-3}(A_2,f,w;x)$	~~Dini-Lip.~~ on [-1,1] [21]				Dini-Lip. on [-1,1] [21]
$H^{\circ}_{2n-1}(f,w;x)$	C[-1,1] [8]	$\alpha,\beta < 0$: C[-1,1] [23]	α or $\beta \geq 0$ $C[-1,1]^{(*)}$ [23]	[1], [2]	C[-1,1] [19]
$\tilde{H}^{\circ}_{2n-1}(f,w;x)$				[3], [20]	
$H_{2n-1}(f,w;x)$	C[-1,1] [8]	$\alpha,\beta < 0$: C[-1,1] [10]	α or $\beta \geq 0$: $C[-1,1]^{(*)}$ [23]		

$^{(*)}$Convergence to $f(x)$ is uniform only on $[a,b] \subset (-1,1)$.

7. Summary of Thesis.

This work has been motivated by the results of Motzkin, Sharma and Straus and Saxena and Sharma on averaging interpolation, in the hope that the averaging Hermite interpolators (introduced in Chapter I), might have better convergence properties than the ordinary Hermite-Féjer interpolators $H^{\circ}_{2n-1}(f,x)$.

In Chapter I we define the averaging Hermite interpolators, denoted by $H_N(A_m,f;x)$ (sometimes $H_N(A,f;x)$ only). These are polynomials of degree $\leq N = (s+1)n-m-1$ which, at the nodes E_n, interpolate a given function $f(x)$, have assigned derivatives up to order $s-1$, and whose s-th derivatives at the nodes have prescribed averages. When the derivatives are zero, we use the notation $H^{\circ}_N(A_m,f;x)$. We also show that $H_N(A_m,f;x)$ have approximation-theoretic properties analogous to those of the averaging interpolators $L_\nu(A_m,f;x)$ of §§4 and 5. The germ of this idea is contained in a paper of Motzkin and Sharma ([15], Theorem 4), who considered point functionals instead of averages of values. However, it is interesting to note that while the approximation-theoretic properties of $L_\nu(A_m,f;x)$ are found when $A_m(z)$ has negative roots, the corresponding one of $H_N(A_m,f;x)$ is found when the zeros of $A_m(z)$ have the sign of $(-1)^s$, s being the highest derivative used to define $H_N(A_m,f;x)$.

After the introduction of $H_N(A_m,f;x)$, the natural question arises about their convergence as n tends to infinity. We attend to such problems for $H^{\circ}_{2n-2}(A_1,f;x)$ and $H^{\circ}_{2n-3}(A_2,f;x)$, in the particular case when $A_1(z) = 1-z$ and $A_2(z) = 1-2z+z^2$.

In Chapter II we consider the nodes E_n which are the zeros of $(1-x^2)T_{n-2}(x)$ and examine the convergence behaviour of $H^\circ_{2n-1}(f,x)$, $H^\circ_{2n-2}(A_1,f;x)$ and $H^\circ_{2n-3}(A_2,f;x)$. We obtain the necessary and sufficient conditions for the uniform convergence of $H^\circ_{2n-1}(f,x)$, supplementing some recent results of Berman ([1], [2]) who considered mainly the three functions x, $|x|$ and x^2. We then give sufficient conditions for the uniform convergence of $H^\circ_{2n-2}(A_1,f;x)$, $H^\circ_{2n-3}(A_2,f;x)$, and show that each of these two operators converges uniformly for a larger class of functions than $H^\circ_{2n-1}(f,x)$. We also show that, in this sense, $H^\circ_{2n-3}(A_2,f;x)$ is better than $H^\circ_{2n-2}(A_1,f;x)$.

Chapter III is devoted to the study of the convergence behaviour of $H^\circ_{2n-2}(A_1,f;x)$, $H^\circ_{2n-3}(A_2,f;x)$ and of the averaging interpolators $L_{n-3}(A_2,f;x)$ when E_n consists of the zeros of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, $\alpha,\beta > -1$. We also consider the Grünwald-type mean of $L_{n-3}(A_2,f;x)$. The results obtained are analogous to those of Vertesi [26], Grünwald [10] and Szegö [23].

In Chapter IV, we study the convergence behaviour of the trigonometric averaging interpolators $T_{n-1}(A_2,f;x)$ on equidistant nodes, in the case when $A_2(z) = (1+rz)(1+sz)$, for all real r,s.

CHAPTER II

AVERAGING INTERPOLATION ON SETS WITH MULTIPLICITIES

1. Introduction.

Given n points $x_1 < \ldots < x_n$ and a polynomial $A_m(z) = \sum_{k=o}^{m} a_k z^k$ with negative roots, it is known ([17], Theorem 1, p. 196) that there exists a unique polynomial $P(x)$ of degree $\leq n-m-1$ which satisfies the interpolatory conditions (4.2) of Chapter 0, for any given vector $y = (y_1,\ldots,y_n)$. In addition, such a $P(x)$ minimizes certain m functionals, in general distinct, over all $Q \in \Pi_{n-m-1}$.

In this chapter, we generalize the above result to the case in which each point x_h, $h = 1,\ldots,n$ has a fixed multiplicity s. The main interpolation theorem is contained in §2; in §3 we outline a method for constructing the interpolatory polynomials and in §4 we obtain explicit formulae for two special cases. In §5 we study the case of the particular point sets $x_{h+1} = \beta x_h + \gamma$, $h = 1,\ldots,n-1$, thus generalizing some of the results of [17]. In §6 we apply the results of §5 to trigonometric averaging interpolation. Finally, in §7, we shall exhibit the partial sums $I_{n,n-q}(f,x)$ of the trigonometric interpolators on equidistant nodes as averaging interpolators.

It may be remarked that Theorems 2.1 and 2.2 below give for $s = 0$ simplified and revised versions of Theorems 2 and 3 of [17].

2. Averaging Hermite Interpolators.

Let $x_1 < \dots < x_n$ be n given real numbers and let $p = (p_1,\dots,p_n)$ be a given vector with positive components. If $m \leq n-1$ is a given integer, and the α_ℓ are given numbers, set

$$(2.1)\quad \begin{aligned} A_m(z) &= \prod_{\ell=1}^{m} (1+\alpha_\ell z) = \sum_{k=o}^{m} a_k z^k \\ A_{m\ell}(z) &= A_m(z)/(1+\alpha_\ell z) = \sum_{k=o}^{m-1} a_{k\ell} z^k, \quad \ell = 1,\dots,m \end{aligned}$$

and consider the linear functionals

$$(2.2)\quad \begin{aligned} L_j(y) &= A_m(E)(p_j y_j) = \sum_{k=o}^{m} a_k p_{k+j} y_{k+j}, \quad j = 1,\dots,n-m \\ L_{j\ell}(y) &= A_{m\ell}(E)(p_j y_j) = \sum_{k=o}^{m-1} a_{k\ell} p_{k+j} y_{k+j}, \quad j = 1,\dots,n-m+1; \\ &\qquad \ell = 1,\dots,m, \end{aligned}$$

where E is the usual forward shift operator on sequences and y is either an n-vector or a function, in which case $y_h = y(x_h)$, $h = 1,\dots,n$. Since $A_m(E) = (1+\alpha_\ell E)A_{m\ell}(E)$, it is easy to see that

$$(2.3)\quad L_j(y) = L_{j\ell}(y) + \alpha_\ell L_{j+1,\ell}(y), \quad j = 1,\dots,n-m; \quad \ell = 1,\dots,m.$$

The problem of averaging interpolation is as follows:

Problem. Given s n-vectors $y^{(0)},\ldots,y^{(s)}$ $(s \geq 0)$, find a polynomial $H(x) = H_N(A_m,y;x)$ of degree $N = (s+1)n-m-1$ (abbreviated $H \in \Pi_N$) satisfying

$$
(2.4) \qquad \begin{cases} H^{(i)}(x_h) = y_h^{(i)} , & i = 0,\ldots,s-1; \quad h = 1,\ldots,n \\ \\ L_j(H^{(s)}) = L_j(y^{(s)}), & j = 1,\ldots,n-m \ . \end{cases}
$$

Theorem 2.1. *Let* $s \geq 0$ *be a given integer and let* $\alpha_1,\ldots,\alpha_m$ *of* (2.1) *be such that* $(-1)^s\alpha_\ell > 0$, $\ell = 1,\ldots,m$. *Then there exists a unique polynomial* $H \in \Pi_N$, $N = (s+1)n-m-1$, *satisfying* (2.4). *Furthermore*, $H(x)$ *minimizes each of*

$$
(2.5) \qquad \max_{j=1,\ldots,n-m+1} |\alpha_\ell^{j-1} L_{j\ell}(P^{(s)}-y^{(s)})| , \quad \ell = 1,\ldots,m
$$

over all polynomials $P \in \Pi_N$ *for which*

$$
(2.6) \qquad P^{(i)}(x_h) = y_h^{(i)} , \quad i = 0,\ldots,s-1; \quad h = 1,\ldots,n \ .
$$

Proof. Clearly (2.4) has a unique solution for every $y^{(0)},\ldots,y^{(s)}$ if and only if $y^{(i)} \equiv 0$, $i = 0,\ldots,s$, implies $H(x) \equiv 0$. If $H \in \Pi_N$ satisfies (2.4) with $y^{(i)} \equiv 0$, $i = 0,\ldots,s$, then

$$
(2.7) \qquad H(x) = (w(x))^s R(x) , \quad w(x) = (x-x_1),\ldots,(x-x_n) ,
$$

where $R(x)$ is continuous (here, in particular, $R \in \Pi_{n-m-1}$), and hence

$$H^{(s)}(x_h) = s!(w'(x_h))^s R(x_h) \;, \qquad h = 1,\dots,n \;.$$

Applying (2.3) to the last n-m equations of (2.4), we obtain

$$0 = L_j(H^{(s)}) = L_{j1}(H^{(s)}) + \alpha_1 L_{j+1,1}(H^{(s)}) \;, \qquad j = 1,\dots,n-m \;. \tag{2.8}$$

We now show that R(x) has at least n-m zeros which, by (2.7), implies that H(x) has at least N+1 zeros, and hence $H(x) \equiv 0$.

If s is even, $\alpha_1,\dots,\alpha_m$ are positive, and (2.8) shows that the sequence

$$\{L_{j1}(H^{(s)})\}_{j=1}^{n-m+1} = \{A_{m1}(E)(s!p_j(w'(x_j))^s R(x_j))\}_{j=1}^{n-m+1}$$

has n-m sign changes. Since all the numbers α_ℓ and $s!p_j(w'(x_j))^s$ are positive, it is easy to see that the sequence $\{L_{j1}(H^{(s)})\}_{j=1}^{n-m+1}$ has no more sign changes than $\{R(x_j)\}_{j=1}^{n}$ ([17], Theorem 1, p. 196). Hence R(x) has at least n-m zeros.

If s is odd, $\alpha_1,\dots,\alpha_m$ are negative and sgn $(w'(x_h))^s = (-1)^{n-h}$, $h = 1,\dots,n$. Because of the alternating character of the sequence $w'(x_j)$, we observe that

$$L_{j1}(H^{(s)} = (1+\alpha_2 E)\dots(1+\alpha_m E)((-1)^{n-j} s!p_j|w'(x_j)|^s R(x_j)) =$$

$$= (-1)^{n-j}(1-\alpha_2 E)\dots(1-\alpha_m E)(s!p_j|w'(x_j)|^s R(x_j)) \;,$$

$$j = 1,\dots,n-m+1$$

and as the numbers $-\alpha_\ell$ and $s!p_j|w'(x_j)|^s R(x_j)$ are positive, we can apply the same argument as before to show that also in this case $R(x)$ has at least $n-m$ zeros.

In order to prove that $H(x)$ minimizes (2.5) let us observe that, applying (2.3) to (2.4), we obtain for $\ell = 1,\dots,m$, the $n-m-1$ equalities

$$L_{1\ell}(H^{(s)}-y^{(s)}) = -\alpha_\ell L_{2\ell}(H^{(s)}-y^{(s)}) =$$

$$= (-\alpha_\ell)^2 L_{3\ell}(H^{(s)}-y^{(s)}) = \dots = \rho_\ell .$$

Now, let us suppose that there is a $P \in \Pi_N$ satisfying (2.6) and such that, for some $\ell = 1,\dots,m$,

$$\max_{j=1,\dots,n-m-1} |\alpha_\ell^{j-1} L_{j\ell}(P^{(s)}-y^{(s)})| \le |\rho_\ell|$$

Then $\pm(-\alpha_\ell)^{j-1} L_{j\ell}(H^{(s)}-P^{(s)}) \ge 0$, $j = 1,\dots,n-m+1$ (+ if $\rho_\ell \ge 0$, – if $\rho_\ell < 0$) hence, if s is even, $\{L_{j\ell}(H^{(s)}-P^{(s)})\}_{j=1}^{n-m+1}$ has $n-m$ sign changes. Also, $H(x) - P(x) \in \Pi_N$ and satisfies

$$H^{(i)}(x_h) - P^{(i)}(x_h) = 0 , \quad i = 0,\dots,s-1; \; h = 1,\dots,n .$$

Hence, as above, $H(x) \equiv P(x)$. The same argument holds if s is odd, so that Theorem 2.1 is completely proved. □

An analogous result holds for trigonometric polynomials $T(x)$ of degree M (abbreviated $T \in \Pi_M^*$).

Theorem 2.2. *Let* $s \geq 0$ *be a given integer, let* $\alpha_1,\ldots,\alpha_m$ *of* (2.1) *be such that* $(-1)^s\alpha_\ell > 0$, $\ell = 1,\ldots,m$ *and assume that* $x_n - x_1 < 2\pi$ *and that* $N = (s+1)n-m-1$ *is an even number*, $N = 2M$. *Then there exists a unique* $T \in \Pi_M^*$ *satisfying* (2.4). *Furthermore*, $T(x)$ *minimizes each of*

$$\max_{j=1,\ldots,n-m+1} |\alpha_\ell^{j-1} L_j(U^{(s)}-y^{(s)})| , \qquad \ell = 1,\ldots,m$$

over all $U \in \Pi_M^*$ *for which*

$$U^{(i)}(x_h) = y_h^{(i)} , \qquad i = 0,\ldots,s-1; \quad h = 1,\ldots,n .$$

The proof is like that of Theorem 2.1. Note that the conditions $x_n - x_1 < 2\pi$, $N = 2M$ are necessary in order to deduce $H(x) \equiv 0$ from the fact that $H(x)$ has at least $N+1$ zeros on $[x_1,x_n]$.

3. A Method for Finding $H(x)$.

All the previous notations are retained and, for simplicity, we shall write H_j for $H(x_j)$, $H_j^{(s)}$ for $H^{(s)}(x_j)$, etc.

(i): $s = 0$. Following the method of ([17], pp. 199-200), set

$$g_j = L_j(H) = \sum_{k=o}^{m} a_k p_{k+j} H_{k+j} , \qquad j = 1,\ldots,n-m$$

$$g_h = \sum_{k=o}^{n-h} a_k p_{k+h} H_{k+h} , \qquad h = n-m+1,\ldots,n ,$$

so that by (2.4) we have $g_j = L_j(y)$, $j = 1,\dots,n-m$ while $g_{n-m+1},\dots,g_n$ are determined by the supposition that H satisfies (2.4). If $1/A_m(z)$ has the expansion

$$1/A_m(z) = \sum_{k=o}^{\infty} b_k z^k ,$$

we obtain after some manipulations

$$p_j H_j = \sum_{k=o}^{n-j} b_k g_{k+j} , \qquad j = 1,\dots,n . \tag{3.1}$$

Since $H \in \Pi_{n-m-1}$, the divided differences of order $n-m$ of the numbers $H_1,\dots,H_n$ must vanish, i.e.,

$$\sum_{k=o}^{n-m} H_{j+k}/w_j'(x_{j+k}) = 0 , \qquad j = 1,\dots,m , \tag{3.2}$$

where $w_j(x) = (x-x_j)\dots(x-x_{n+n-m})$. From (3.1) and (3.2) we obtain after some simplification the following system in the unknowns $g_{n-m+1},\dots,g_n$:

$$\sum_{k=o}^{n-j} c_{kj} g_{k+j} = 0 , \qquad j = 1,\dots,m , \tag{3.3}$$

where

$$c_{kj} = \begin{cases} \displaystyle\sum_{i=o}^{n-m} b_{k-i}/p_{i+j}w_j'(x_{i+j}), & k = n-m+1,\dots,n-j \\[2ex] \displaystyle\sum_{i=o}^{k} b_{k-i}/p_{i+j}w_j'(x_{i+j}), & k = 0,\dots,n-m . \end{cases}$$

The system (3.3) enables us to obtain $g_{n-m-1},\ldots,g_n$, after which we obtain $H(x)$ by Lagrange interpolation to the values $H_1,\ldots,H_n$ given by (3.1).

(ii): $s > 0$. This can be reduced to the case $s = 0$ as follows. Let $Q \in \Pi_{sn}$ denote the ordinary Hermite interpolator uniquely determined by the conditions

$$Q^{(i)}(x_h) = y_h^{(i)}, \qquad i = 0,\ldots,s-1; \quad h = 1,\ldots,n, \tag{3.4}$$

whose explicit expression can be found in ([3], pp. 6-7). Since $H(x)$ satisfies (2.4), we can write

$$H(x) = Q(x) + (w(x))^s R(x), \qquad w(x) = (x-x_1)\ldots(x-x_n) \tag{3.5}$$

where $R \in \Pi_{n-m-1}$, so that

$$H^{(s)}(x_h) = Q^{(s)}(x_h) + s!(w'(x_h))^s R(x_h), \qquad h = 1,\ldots,n. \tag{3.6}$$

Substituting (3.6) into the last $n-m$ equations of (2.4) we obtain

$$L_j(y^{(s)}) = L_j(Q^{(s)}) + s!L_j((w')^s R), \qquad j = 1,\ldots,n-m$$

and hence

$$L_j((w')^s R) = L_j(y^{(s)} - Q^{(s)})/s!, \qquad j = 1,\ldots,n-m$$

which reduces to the case $s = 0$ by setting $p_j = |w'(x_j)|^s$ (when s is odd, $A(z) = \Pi(1+\alpha_\ell z)$ must be replaced by $B(z) = \Pi(1-\alpha_\ell z)$).

4. The Case of $A_1(z) = 1-z, \quad A_2(z) = 1-2z+z^2$.

The averaging Hermite interpolators of this section can be constructed by the method outlined above in §3. However when $A_1(z) = 1-z$ or $A_2(z) = 1-2z+z^2$ it is easier to use a different method.

Let $H^\circ_{2n-1-m}(A_m,y;x)$ denote the averaging Hermite interpolator, relative to the polynomial $A_m(z) = \sum_0^m a_i z^i$, when $s = 1$, $p_k = 1$, $y_k^{(1)} = 0$, $k = 1,\dots,n$ (in §3). Also, let $h_k(x)$ and $h_k^*(x)$ denote the fundamental polynomials of Hermite interpolation ((2.3), (2.4) Chapter I). Set $s_n = x_1 + \dots + x_n$, and

$$(4.1)\quad \left\{ \begin{aligned} J_n &= \sum_{k=1}^n \frac{1}{(w'(x_k))^2} , \quad J_n^* = \sum_{k=1}^n \frac{2s_n - x_k}{(w'(x_k))^2} , \\ K_n &= \sum_{k=1}^n \frac{k}{(w'(x_k))^2} , \quad K_n^* = \sum_{k=1}^n \frac{k(2s_n - x_k)}{(w'(x_k))^2} , \\ F_n &= \sum_{k=1}^n f(x_k) \frac{w''(x_k)}{(w'(x_k))^3} , \end{aligned} \right.$$

$$F_n^* = \sum_{k=1}^n f(x_k) \frac{1+(2s_n - x_k) w''(x_k)/w'(x_k)}{(w'(x_k))^2} .$$

Theorem 4.1. If $m = 1$ and $A_1(z) = 1-z$, then

$$(4.2)\quad \left\{ \begin{aligned} & H^\circ_{2n-2}(A_1,y;x) = \sum_{k=1}^n y_k h_k(x) + c \sum_{k=1}^n h_k^*(x) \\ & c = c_n = F_n/J_n , \end{aligned} \right.$$

where F_n, J_n *are given by* (4.1).

Theorem 4.2. *If* $m = 2$ *and* $A_2(z) = 1-2z+z^2$, *then*

$$(4.3) \qquad H^\circ_{2n-3}(A_2,y;x) = \sum_{k=1}^{n} y_k h_k(x) + \sum_{k=1}^{n} (d+ke)h_k^*(x)$$

where

$$(4.4) \qquad d + ke = H^{\circ\prime}_{2n-3}(A_2,y;x_k) \;, \quad k = 1,\dots,n$$

and d, e *are given by*

$$(4.5) \qquad d \equiv d_n = \frac{-K_nF_n^* + K_n^*F_n}{J_nK_n^* - J_n^*K_n} \;, \qquad e \equiv e_n = \frac{J_nF_n^* - J_n^*F_n}{J_nK_n^* - J_n^*K_n} \;.$$

The proof of Theorem 4.1 being similar to that of Theorem 4.2, we shall only give the proof of Theorem 4.2.

Proof of Theorem 4.2. By definition, $H^{\circ\prime}_{2n-3}(A_2,f)$ satisfies the linear difference equation

$$H^{\circ\prime}_{2n-3}(A_2,y;x_k) - 2H^{\circ\prime}_{2n-3}(A_2,y;x_{k+1}) + H^{\circ\prime}_{2n-3}(A_2,y;x_{k+2}) = 0$$

whose general solution is given by (4.4), with d, e arbitrary. Since $H^\circ_{2n-3}(A_2,y;x)$ must have degree $\leq 2n-3$, we can determine d, e by requiring that the coefficients of x^{2n-1}, x^{2n-2} in $H^\circ_{2n-3}(A_2,y;x)$ vanish. Since

$$(4.6) \qquad H^\circ_{2n-3}(A_2,y;x) = \sum_{k-1}^{n} y_k h_k(x) + \sum_{k=1}^{n} H^{\circ\prime}_{2n-3}(A_2,y;x_k)h_k^*(x) \quad ,$$

by (2.3) and (2.4) of Chapter I, such coefficients are given respectively by

$$dJ_n + eK_n - F_n = 0$$

$$dJ^*_n + eK^*_n - F^*_n = 0$$

whence (4.5). □

Remark 4.1. An interesting property of $H^\circ_{2n-2}(A_1,y;x)$, $H^\circ_{2n-3}(A_2,y;x)$ is that, unlike the ordinary Hermite interpolator $H^\circ_{2n-1}(y,x)$ they reproduce polynomials of degree ≤ 1, as seen easily from the defining equations. If $x_1,\ldots,x_n$ are equidistant, $H^\circ_{2n-3}(A_2,y,x)$ reproduces polynomials of degree ≤ 2.

Remark 4.2. When $x_1,\ldots,x_n$ are symmetrical (i.e., $x_k = -x_{n+1-k}$, $k = 1,\ldots,n$), then $h_k(x) = h_{n+1-k}(-x)$, $h^*_k(x) = -h^*_{n+1-k}(x)$ and it is easy to see that $H^\circ_{2n-1}(A_2,y;x)$, $H^\circ_{2n-2}(A_1,y;x)$, $H^\circ_{2n-3}(A_2,y;x)$ are all even (odd) when $y = y(x)$ is even (odd). Hence, comparing their degrees and their defining equations we can see that

$$H^\circ_{2n-2}(A_1,y;x) = H^\circ_{2n-1}(y,x) \quad \textit{if} \quad y(x) \ \textit{is even},$$

$$H^\circ_{2n-3}(A_2,y;x) = H^\circ_{2n-2}(A_1,y;x) \quad \textit{if} \quad y(x) \ \textit{is odd}.$$

5. Special Point Sets.

If the numbers (not necessarily real) $x_1,\ldots,x_n$ satisfy

(5.1) $$x_{h+1} = \beta x_h + \gamma, \qquad h = 1,\ldots,n-1$$

and $p = (p_1,\ldots,p_n)$ is the special n-vector specified below, we obtain an existence and uniqueness theorem for $H(x)$, more complete than Theorem 2.1 and which generalizes a theorem of ([17], Theorem 4, p. 207). As in [17] we require that

$$(5.2)\qquad \begin{cases} (\beta-1)x_1+\gamma\neq 0 \\ \beta^r\neq 1\ , \quad r=2,\dots,n-1\ \ (\text{if } \beta\neq 1)\ , \end{cases}$$

so that $x_1,\dots,x_n$ are distinct.

Theorem 5.1. Let $x_1,\dots,x_n$ *satisfy* (5.1), (5.2) *and let* $A_m(z)=\sum_{k=o}^{m}a_kz^k$ *where* $m\leq n-1$, *be given. Let* $s\geq 0$ *be a given integer, and let* $p=(p_1,\dots,p_n)$ *be the n-vector with components*

$$(5.3)\qquad p_h=(w'(x_h))^{-s}\ , \qquad h=1,\dots,n\ ,$$

where $w(x)=(x-x_1)\dots(x-x_n)$. *Then there exists a unique polynomial* $H\in\Pi_N$, $N=(s+1)n-m-1$, *satisfying* (2.4), *if and only if*

$$(5.4)\qquad A_m(\beta^r)\neq 0, \qquad r=0,\dots,n-m-1\ .$$

Here $L_1,\dots,L_{n-m}$ *are as in* (2.2), *with* $p_1,\dots,p_n$ *given by* (5.3).

Proof. For $s=0$ the theorem has been proved in ([17], Theorem 4, p. 207, Remark 1, p. 210). If $s>0$ we write

$$H(x)=Q(x)+w(x)^sR(x), \quad w(x)=(x-x_1)\dots(x-x_n)$$

where $Q\in\Pi_{sn}$ satisfies (3.4) and $R\in\Pi_{n-m-1}$ has to be determined. As in §3, we obtain the system of equations

$$L_j((w')^s R) = L_j(y^{(s)} - Q^{(s)})/s!\,, \qquad j = 1,\dots,n-m$$

which is

$$\sum_{k=o}^{m} a_k R(x_{k+j}) = \sum_{k=o}^{m} a_k y_{k+j}, \qquad j = 1,\dots,n-m\,,$$

where

$$y_h = (y_h^{(s)} - q_h^{(s)})/s!\,w'(x_h)^s\,, \qquad h = 1,\dots,n\,.$$

Therefore we have reduced the case $s > 0$ to the case $s = 0$ and the theorem is completely proved. $\square$

6. Trigonometric Analogue.

A result similar to the above holds for trigonometric polynomials when $s = 0$, $m = 2q$.

Theorem 6.1. *Let* $A_m(z) = \sum_{k=o}^{2q} a_k z^k$ *be given. Consider the* $2n+1$ *points* $x_1,\dots,x_{2n+1}$ *given by*

$$x_h = hc, \qquad h = 1,\dots,2n+1, \qquad nc < \pi$$

and let $s = 0$ *and* $p = (1,\dots,1)$. *Then there exists a unique trigonometric polynomial* $T \in \Pi^*_{n-q}$ *satisfying* (2.4), *with* $s = 0$, *if and only if*

(6.1) $$A_m(e^{ic(2q-2n+r)}) \neq 0\,, \qquad r = 0,\dots,2n-2q\,.$$

Proof. Every trigonometric polynomial $T \in \Pi_{n-q}^{*}$ can be written as

$$T(x) = z^{2q-2n}H(z) \;, \qquad z = e^{ix} \;,$$

where $H \in \Pi_{2n-2q}$. Consequently, (2.4) may be written as

$$(6.2) \qquad L_j(T) = \sum_{k=0}^{2q} a_k z_{k+j}^{2q-2n} H(z_{k+j}) = L_j(y) \;, \quad j = 1,\ldots,2n-2q$$

where the points

$$z_h = e^{ihc} \;, \qquad h = 1,\ldots,2n+1$$

satisfy a relation like (4.1) with $\beta = e^{ic}$, $\gamma = 0$. Since $z_{k+j} = z_k z_j$ we can rewrite (6.2) as

$$\sum_{k=0}^{2q} a_k z_k^{2q-2n} H(z_{k+j}) = z_j^{2n-2q} L_j(y) =$$

$$= \sum_{k=0}^{2q} a_k z_k^{2q-2n} (y_{k+j} z_{k+j}^{2n-2q}) \;.$$

Now, this reduces to Theorem 5.1, where $s = 0$ and $A_m(z)$ is replaced by the polynomial

$$A_m^*(z) = \sum_{k=0}^{2q} (a_k z_k^{2q-2n}) z^k = A_m(z e^{ic(2q-2n)}) \;.$$

Since condition (5.4) becomes $A_m^*(e^{irc}) \neq 0$, $r = 0,\ldots,2n-2q$, we obtain (6.1). □

7. The Polynomials $I_{n,n-q}(f,x)$.

Zygmund ([28], p. 8) has introduced the partial sums

$$(7.1)\quad \begin{cases} I_{n,n-q}(f,x) = \frac{1}{2} a_0 + \sum_{k=1}^{n-q} (a_k \cos kx + b_k \sin kx), \quad q = 1,\dots,n-1 \\ a_k = a_{kn}(f) = \frac{2}{2n+1} \sum_{j=o}^{2n} f(x_j) \cos kx_j \ , \\ b_k = b_{kn}(f) = \frac{2}{2n+1} \sum_{j=o}^{2n} f(x_j) \sin kx_j \ , \end{cases}$$

of the trigonometric polynomial $I_n(f,x) \in \Pi_n^*$ interpolating to a 2π-periodic function $f(x)$ on the points $x_1,\dots,x_{2n+1}$ given by

$$(7.2)\qquad x_{h+1} = \frac{2h\pi}{2n+1}\ , \qquad h = 0,1,\dots,2n\ .$$

For the polynomials $I(x) = I_{n,n-q}(f,x) \in \Pi_{n-q}^*$ we have

Theorem 7.1. For every $0 \le q \le n-1$, $I(x)$ is an averaging interpolator in the sense of Theorem 4.2. More precisely, we have

$$(7.3)\qquad L_j(I) = L_j(f)\ , \qquad j = 1,\dots,2n-2q\ ,$$

where

$$L_j(f) = \sum_{k=o}^{2q} a_k f(x_{k+j})\ , \qquad a_k = \prod_{\ell=2q+1}^{2n} \sin \frac{k-\ell}{2n+1}\pi\ .$$

Proof. It is known ([28], Vol. 2, p. 8) that $I(x) = I_{n,n-q}(f,x)$ minimizes

$$\sum_{k=o}^{2n} (f(x_k)-s(x_k))^2$$

over all trigonometric polynomials $s \in \Pi^*_{n-q}$; hence by a familiar argument, we see that

$$(7.4) \qquad \sum_{k=o}^{2n} s(x_k)(f(x_k)-I(x_k)) = 0 , \qquad s \in \Pi^*_{n-q} .$$

In particular, (7.4) will hold for each of the trigonometric polynomials

$$(7.5) \qquad s_j(x) = \prod_{\ell=o}^{2n} \sin \frac{1}{2} (x-x_\ell) \Big/ \prod_{\ell=j}^{2q+j} \sin \frac{1}{2} (x-x_\ell), \qquad j = 0,\ldots,2n-2q .$$

Since $s_j(x_k)$ is nonzero only for $k = j,\ldots,2q+j$, on substituting (7.5) into (7.4) we obtain

$$(7.6) \qquad 0 = \sum_{k=j}^{2q+j} s_j(x_k)(f(x_k)-I(x_k)) = \sum_{k=o}^{2q} s_j(x_{k+j})(f(x_{k+j})-I(x_{k+j})) , \qquad j = 0,\ldots,2n-2q .$$

Now, it is easy to see that $s_j(x_{k+j}) = s_0(x_k) = a_k$, hence (7.6) reduces to (7.3) and the theorem is proved. □

CHAPTER III

CONVERGENCE PROPERTIES OF HERMITE AND AVERAGING HERMITE INTERPOLATORS ON EXTENDED CHEBYSHEV NODES

1. Introduction.

In this chapter we study the convergence properties of the Hermite-Féjer interpolators $H^\circ_{2n+3}(f,x)$ (with zero derivatives at the nodes) and the averaging Hermite interpolators relative to the polynomials

$$A_1(z) = 1-z \ , \quad A_2(z) = 1-2z+z^2 \ , \tag{1.0}$$

which we denote respectively by

$$H^\circ_{2n+2}(A_1,f;x) \ , \qquad H^\circ_{2n+1}(A_2,f;x) \ ,$$

based on the extended Chebyshev nodes $x_o,\dots,x_{n+1}$:

$$\begin{aligned} x_o &= x_{o,n+2} = 1, \qquad x_{n+1} = x_{n+1,n+2} = -1 \ , \\ x_k &= x_{k,n+2} = \cos\frac{2k-1}{2n}\pi \ , \quad k = 1,\dots,n \ , \end{aligned} \tag{1.1}$$

which are the zeros of $w(x) = (1-x^2)T_n(x)$, where $T_n(x) = \cos n \arccos x$.

In §2, we obtain the explicit form of $H^\circ_{2n+2}(A_1,f;x)$ and $H^\circ_{2n+1}(A_2,f;x)$. In §3, we find some necessary and sufficient conditions for the uniform convergence of $H^\circ_{2n+3}(f,x)$ to $f(x)$ on $1 \geq x \geq -1$. As a corollary we find that if $f(x)$ has a nonzero derivative at $x = 1$ (or $x = -1$), then $H^\circ_{2n+3}(f,x)$ does not converge uniformly to $f(x)$, thus improving earlier results of Saxena [18] and D.L. Berman [1]. In §§4 and 5, we obtain some sufficient conditions for the uniform convergence of $H^\circ_{2n+2}(A_1,f;x)$ and $H^\circ_{2n+1}(A_2,f;x)$ to $f(x)$ on $1 \geq x \geq -1$. Comparing the results obtained in §§3, 4 and 5, we find in §6 that $H^\circ_{2n+2}(A_1,f;x)$ and $H^\circ_{2n+1}(A_2,f;x)$ are, in some sense, better than $H^\circ_{2n+3}(f,x)$. For example the class of functions $f(x)$ such that $H^\circ_{2n+2}(A_1,f;x) \to f(x)$ uniformly is larger than the class of functions $f(x)$ such that $H^\circ_{2n+3}(f,x) \to f(x)$ uniformly.

2. Explicit Form of $H^\circ_{2n+2}(A_1,f;x)$ and $H^\circ_{2n+1}(A_1,f;x)$.

These polynomials can be obtained by replacing, in (4.1) of Chapter II, n with n+2, the index of summation k with k-1 and taking sums from 0 to n+1.

Since $T_n(x)$ satisfies the equation

$$(1-x^2)T_n'' - xT_n' + n^2T_n = 0 \quad ,$$

it is easy to see that, if $w(x) = (1-x^2)T_n(x)$, then

$$\frac{w''(x_k)}{w'(x_k)} = \begin{cases} -\dfrac{3x_k}{1-x_k^2} \;, & x_k \neq \pm 1 \\ \pm(2n^2+1) \;, & x_k = \pm 1 \;, \end{cases}$$

$$\frac{1}{(w'(x_k))^2} = \begin{cases} \dfrac{1}{n^2(1-x_k^2)} \;, & x_k \neq \pm 1 \\[2ex] \dfrac{1}{4} \;, & x_k = \pm 1 \;. \end{cases}$$

Since the points (1.1) are symmetrical, we have $s_{n+2} \equiv x_o + \ldots + x_{n+1} = 0$. It is easy to see, using the identity

$$\text{(2.1)} \qquad \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{1-x_k^2} = \frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{1-x_k} = 1$$

that (4.1) of Chapter II becomes, after simplification,

$$\text{(2.2)} \quad \begin{cases} J_{n+2} = \displaystyle\sum_{k=o}^{n+1} \frac{1}{(w'(x_k))^2} = \frac{3}{2} \;, \qquad J^*_{n+2} = \sum_{k=o}^{n+1} \frac{-x_k}{(w'(x_k))^2} = 0 \;, \\[3ex] K_{n+2} = \displaystyle\sum_{k=o}^{n+1} \frac{k+1}{(w'(x_k))^2} = \frac{n+2}{2} , \\[3ex] K^*_{n+2} = \displaystyle\sum_{k=o}^{n+1} \frac{-(k+1)x_k}{(w'(x_k))^2} = \frac{1}{n^2} \sum_{k=1}^{n} \frac{kx_k}{1-x_k^2} + \frac{n}{4} \;, \\[3ex] F_{n+2} \equiv F_{n+2}(f) = \displaystyle\sum_{k=o}^{n+1} f(x_k) \frac{w''(x_k)}{(w'(x_k))^3} = \\[3ex] \qquad = \displaystyle -\frac{3}{n^2} \sum_{k=1}^{n} \frac{x_k f(x_k)}{(1-x_k^2)^2} + \frac{n^2}{2} \, [f(1)-f(-1)] \;, \\[3ex] F^*_{n+2} \equiv F^*_{n+2}(f) = \displaystyle\sum_{k=o}^{n+1} \frac{f(x_k)}{(w'(x_k))^2} - \sum_{k=o}^{n+1} x_k f(x_k) \frac{w''(x_k)}{(w'(x_k))^3} \;, \\[3ex] \qquad = \displaystyle\frac{1}{n^2} \sum_{k=1}^{n} \frac{f(x_k)}{1-x_k^2} + \frac{1}{4} \, [f(1)-f(-1)] - F_{n+2}(xf) \;. \end{cases}$$

If we denote by $h_k(x)$, $h_k^*(x)$, $k = 0,\dots,n+1$ the fundamental polynomials of Hermite interpolation at the points (1.1), then (4.5) of Chapter I is replaced by

$$d_{n+2} = \frac{-K_{n+2}\,F_{n+2}^* + K_{n+2}^*\,F_{n+2}}{J_{n+2}\,K_{n+2}^* - J_{n+2}^*\,K_{n+2}} ,$$

$$e_{n+2} = \frac{J_{n+2}\,F_{n+2}^* - J_{n+2}^*\,F_{n+2}}{J_{n+2}\,K_{n+2}^* - J_{n+2}^*\,K_{n+2}} .$$

We see, by (2.2), that (4.2), (4.3) and (4.4) of Chapter II become

$$(2.3)\qquad \begin{cases} H^\circ_{2n+2}(A_1,f;x) = \sum\limits_{k=o}^{n+1} f(x_k)\,h_k(x) + c_{n+2}\sum\limits_{k=o}^{n+1} h_k^*(x) , \\ c_{n+2} = F_{n+2}/J_{n+2} . \end{cases}$$

and

$$(2.4)\qquad H^\circ_{2n+1}(A_2,f;x) = \sum_{k=o}^{n+1} f(x_k)h_k(x) + \sum_{k=o}^{n+1} [d_{n+2}+(k+1)e_{n+2}]h_k^*(x) ,$$

$$(2.5)\qquad d_{n+2} + (k+1)e_{n+2} = H^{\circ\prime}_{2n+1}(A_2,f;x_k), \quad k = 0,\dots,n+1 ,$$

where

$$(2.6)\qquad d_{n+2} = -\frac{n+2}{2}\,e_{n+2} + c_{n+2} , \quad e_{n+2} = F_{n+2}^*/K_{n+2}^* .$$

3. Convergence of $H^\circ_{2n+3}(f,x)$.

We recall that the polynomial $H^\circ_{2n+3}(f,x)$ is determined by

$$(3.1)\qquad H^{\circ}_{2n+3}(f,x_k) = f(x_k)\ ,\qquad H^{\circ\prime}_{2n+3}(f,x_k) = 0,\qquad k = 0,\dots,n+1\ .$$

Let $H^{\circ}_{2n-1}(f,x)$ denote the Hermite interpolator given by

$$(3.2)\qquad H^{\circ}_{2n-1}(f,x_k) = f(x_k),\qquad H^{\circ\prime}_{2n-1}(f,x_k) = 0,\quad k = 1,\dots,n\ .$$

We shall prove:

Theorem 3.1. *If* $f \in C[-1,1]$, *the following three conditions are equivalent*:

$$(3.3)\qquad H^{\circ}_{2n+3}(f,x) \to f(x)\quad \underline{\text{uniformly on}}\quad 1 \geq x \geq -1,$$

$$(3.4)\qquad 2n^2[f(\pm 1) - H^{\circ}_{2n-1}(f,\pm 1)] + H^{\circ\prime}_{2n-1}(f,\pm 1) = o(1)\quad ,$$

$$(3.5)\qquad \frac{1}{n^2}\sum_{k=1}^{n} \frac{f(\pm 1) - f(x_k)}{(1 \mp x_k)^2} = o(1)\quad .$$

Here, (3.4) and (3.5) each consist of two separate conditions at $x = \pm 1$, which must hold simultaneously.

Proof. (i): (3.3) <==> (3.4). Set

$$(3.6)\qquad f_1(x) = \tfrac{1}{2}[f(x)+f(-x)],\qquad f_2(x) = \tfrac{1}{2}[f(x)-f(-x)]\quad ,$$

so that

$$H^{\circ}_{2n+3}(f,x) = H^{\circ}_{2n+3}(f_1,x) + H^{\circ}_{2n+3}(f_2,x)\quad .$$

Since $H^{\circ}_{2n+3}(f,x) = \sum_{k=0}^{n} f(x_k)h_k(x)$ and $h_k(x) = -h_{n+1-k}(-x)$ by the symmetry of the points (1.1), it is easy to see that

$$
(3.7)\qquad
\begin{aligned}
H^{\circ}_{2n+3}(f_1,x) &= \tfrac{1}{2}[H^{\circ}_{2n+3}(f,x) + H^{\circ}_{2n+3}(f,-x)] \\
H^{\circ}_{2n+3}(f_2,x) &= \tfrac{1}{2}[H^{\circ}_{2n+3}(f,x) - H^{\circ}_{2n+3}(f,-x)] \quad .
\end{aligned}
$$

Therefore, it is sufficient to prove the equivalence of (3.3), (3.4) and (3.5) when $f(x) = f_1(x)$ and $f(x) = f_2(x)$. The proof in the two cases being similar, we limit ourselves to the case when $f(x) = f_2(x)$.

By (3.2) and (3.7) it is clear that $H^{\circ}_{2n+3}(f_2,x)$ and $H^{\circ}_{2n-1}(f_2,x)$ are odd. Therefore, by (3.1), (3.2), we can write, for some p_n, q_n,

$$
(3.8)\qquad H^{\circ}_{2n+3}(f_2,x) - H^{\circ}_{2n-1}(f_2,x) = (p_n x + q_n x^3)(T_n(x))^2 \quad .
$$

Since by Féjer's result [8] $H^{\circ}_{2n-1}(f_2,x) \to f_2(x)$ uniformly on $1 \geq x \geq -1$, we see from (3.8) that (3.3) is equivalent to $p_n \to 0$, $q_n \to 0$ simultaneously. Evaluating (3.8) and its derivative at $x = 1$ we obtain, on simplifying by means of (3.1),

$$
\begin{aligned}
p_n + q_n &= f_2(1) - H^{\circ}_{2n-1}(f_2,1) \\
p_n + 3q_n &= -2n^2[f_2(1) - H^{\circ}_{2n-1}(f_2,1)] - H^{\circ\prime}_{2n-1}(f_2,1) \quad .
\end{aligned}
$$

Since by the above remark $p_n + q_n \to 0$, the condition $p_n \to 0$, $q_n \to 0$ is then equivalent to $p_n + 3q_n \to 0$. Hence (3.4) holds since $f_2(1) = -f_2(-1)$.

(ii): (3.4) <==> (3.5). Here we assume $f(x)$ completely arbitrary. It is enough to prove that

$$2n^2[f(1)-H^\circ_{2n-1}(f,1)] + H^{\circ\prime}_{2n-1}(f,1) = o(1)$$

is equivalent to

$$\sum_{k=1}^{n} \frac{f(1) - f(x_k)}{(1-x_k)^2} = o(n^2) \quad ,$$

as the proof for $x = -1$ is quite similar. Differentiating the known formula

$$H^\circ_{2n-1}(f,x) = \frac{1}{n^2} \sum_{k=1}^{n} f(x_k)(1-xx_k) \left\{ \frac{T_n(x)}{x-x_k} \right\}^2$$

and simplifying by means of $T_n(1) = 1$, $T_n'(1) = 0$ we get

$$H^{\circ\prime}_{2n-1}(f,1) = -\frac{3}{n^2} \sum_{k=1}^{n} \frac{f(x_k)}{(1-x_k)^2} \quad ,$$

so that

$$\begin{aligned} &2n^2[f(1)-H^\circ_{2n-1}(f,1)] + H^{\circ\prime}_{2n-1}(f,1) = \\ (3.9) \qquad &= 2\sum_{k=1}^{n} \frac{f(1)}{1-x_k} - \frac{3}{n^2} \sum_{k=1}^{n} \frac{f(x_k)}{(1-x_k)^2} \quad . \end{aligned}$$

When $f(x) \equiv 1$ then $H_{2n-1}(f,x) \equiv 1$ so that the left side of (3.9) vanishes. This yields

$$\sum_{k=1}^{n} \frac{2}{1-x_k} = \frac{3}{n^2} \sum_{k=1}^{n} \frac{1}{(1-x_k)^2} . \tag{3.10}$$

Thus (3.9) can be further simplified as follows:

$$2n^2[f(1)-H^{\circ}_{2n-1}(f,1)] + H^{\circ\prime}_{2n-1}(f,1) =$$

$$= \frac{3}{n^2} \sum_{k=1}^{n} \frac{f(1)-f(x_k)}{(1-x_k)^2} .$$

This completes the proof of Theorem 3.1. □

A more practical sufficient condition for $H^{\circ}_{2n+3}(f)$ to converge uniformly to f is given by:

Theorem 3.2. If $f \in C[-1,1]$ is differentiable at $x = \pm 1$ and

$$f'(1) = f'(-1) = 0 , \tag{3.11}$$

then $H^{\circ}_{2n+3}(f) \to f$ uniformly on $1 \geq x \geq -1$.

Proof. After Theorem 3.1, it is enough to show that (3.11) implies (3.5).

We consider only the case $x = 1$, the proof for $x = -1$ being similar. For arbitrary δ $(0 < \delta < 1)$

$$\left| \sum_{k=1}^{n} \frac{f(1)-f(x_k)}{(1-x_k)^2} \right| \leq \left| \sum_{|1-x_k|>\delta} \right| + \left| \sum_{|1-x_k|\leq\delta} \right| = I_1 + I_2 . \tag{3.12}$$

Since $f'(1)$ exits, we obtain immediately,

$$I_1 \leq \frac{1}{\delta} \sum_{k=1}^{n} \left| \frac{f(1)-f(x_k)}{1-x_k} \right| \leq C \frac{n}{\delta}$$

for some C independent of n, and, since $1 - x_k > 0$,

$$I_2 \le \max_{|1-x_k| \le \delta} \left| \frac{f(1)-f(x_k)}{1-x_k} \right| \sum_{k=1}^{n} \frac{1}{1-x_k} .$$

Using the identity (2.1) and the hypothesis (3.11) we obtain

$$I_1 + I_2 \le C \frac{n}{\delta} + n^2 \epsilon(\delta)$$

where $\epsilon(\delta) \to 0$ if $\delta \to 0$. Taking $\delta = 1/\log n$ and using (3.12) we thus get (3.5). $\Box$

Remark 3.1. Since

$$C_1(\frac{k}{n})^2 \le 1-x_k = 2 \sin^2 \frac{2k-1}{4n} \pi \le C_2(\frac{k}{n})^2 , \quad k = 1,\ldots,n \tag{3.13}$$

where C_1, C_2 are independent of k, n, it is easy to see that if $f \in C[-1,1]$ has a nonzero (finite or infinite) derivative at $x = 1$, then

$$\sum_{k=1}^{n} \frac{f(1)-f(x_k)}{(1-x_k)^2} \ge C_2 n^2 .$$

with C_2 independent of n. Hence, by Theorem 3.1, $H^{\circ}_{2n+3}(f,x)$ does not converge uniformly to $f(x)$. In particular, when $f(x) = x$, $|x|$, or x^2, the Hermite interpolator $H^{\circ}_{2n+3}(f,x)$ with zero derivatives at the extended Chebyshev nodes does not converge uniformly to $f(x)$ (Berman [1], Saxena [18]).

4. Convergence of $H^{\circ}_{2n+2}(A_1,f;x)$.

When $f(x)$ is an even function we have, by Remark 4.2 of Chapter I, $H^{\circ}_{2n+2}(A_1,f;x) = H^{\circ}_{2n+3}(f,x)$, hence the theorem below is a corollary of Theorem 3.1.

Theorem 4.1. *If* $f \in C[-1,1]$ *is even, then* $H^{\circ}_{2n+2}(A_1,f;x) \to f(x)$ *uniformly on* $1 \geq x \geq -1$ *if and only if*

$$\frac{1}{n^2}\sum_{k=1}^{n}\frac{f(1)-f(x_k)}{(1-x_k)^2} = o(1) \quad (4.1)$$

We shall now prove:

Theorem 4.2. *If* $f \in C[-1,1]$ *is odd and satisfies*

$$\frac{1}{n^2}\sum_{k=1}^{n}\frac{f(1)-f(x_k)}{(1-x_k)^2} = o\left(\frac{n}{\log n}\right) \quad (4.2)$$

then $H^{\circ}_{2n+2}(A_1,f;x) \to f(x)$ *uniformly on* $1 \geq x \geq -1$.

Proof. $H^{\circ}_{2n+2}(A_1,f;x)$ is odd by Remark 4.2 of Chapter II hence, if $H_{2n-1}(f,x)$ is the Hermite polynomial defined by

$$H_{2n-1}(f,x_k) = f(x_k), \quad H'_{2n-1}(f,x_k) = H^{\circ\prime}_{2n+2}(A_1,f;x_k), \quad (4.3)$$

$$k = 1,\ldots,n$$

then, by symmetry of (1.1), $H_{2n-1}(f,x)$ is also odd so that, for some p_n,

$$(4.4)\qquad H^{\circ}_{2n+2}(A_1;f;x) - H_{2n-1}(f,x) = p_n x(T_n(x))^2 \;.$$

Setting $x = 1$ and simplifying by using $H^{\circ}_{2n+2}(A_1,f;1) = f(1)$ we obtain

$$(4.5)\qquad p_n = f(1) - H_{2n-1}(f,1) \;.$$

Now, from (4.4) and (4.5) it is clear that $H^{\circ}_{2n+2}(A_1,f;x) \to f(x)$ uniformly on $1 \geq x \geq -1$ is equivalent to

$$(4.6)\qquad H_{2n-1}(f,x) \to f(x) \quad \text{uniformly}, \quad 1 \geq x \geq -1.$$

By Féjer's result, a sufficient condition for (4.6) is

$$(4.7)\qquad H'_{2n-1}(f,x_k) = o\left(\frac{n}{\log n}\right) \;, \quad k = 1,\ldots,n \;.$$

From (4.3), (2.2) and (2.3) we see that

$$(4.8)\qquad H'_{2n-1}(f,x_k) = c_{n+2} = \frac{2}{3} F_{n+2}(f) \;, \quad k = 1,\ldots,n \;.$$

Using (2.2) and observing that since $f(x)$ is odd and the points (1.1) are symmetrical, we get

$$(4.9)\qquad \frac{1}{n^2}\sum_{k=1}^{n} \frac{2x_k f(x_k)}{(1-x_k^2)^2} = \frac{1}{n^2}\sum_{k=1}^{n} \frac{(1+x_k)^2 f(x_k)}{(1-x_k)^2(1+x_k)^2} =$$

$$= \frac{1}{n^2}\sum_{k=1}^{n} \frac{f(x_k)}{(1-x_k)^2} \;.$$

Also, from (2.1) and (3.10) it follows easily that

$$\frac{1}{n^2}\sum_{k=1}^{n}\frac{1}{(1-x_k)^2} = \frac{2n^2}{3} \quad . \tag{4.10}$$

Multiplying (4.10) by $\frac{3}{4}[f(1)-f(-1)] = \frac{3}{2}f(1)$, (4.9) by -3 and adding, we obtain

$$F_{n+2}(f) = \frac{3}{2n^2}\sum_{k=1}^{n}\frac{f(1)-f(x_k)}{(1-x_k)^2} \quad . \tag{4.11}$$

By (4.7) and (4.8), it follows that (4.2) implies (4.6), and hence $H^{\circ}_{2n+2}(A_1,f;x) \to f(x)$ uniformly. □

Corollary 4.1. If $f \in C[-1,1]$ is odd and

$$f(1) - f(x) = o\left(\frac{-\sqrt{1-x}}{\log\,(1-x)}\right), \quad x \to 1- \tag{4.12}$$

then $H^{\circ}_{2n+2}(A_1,f;x) \to f(x)$ uniformly on $1 \geq x \geq -1$.

Proof. As in the proof of Theorem 3.2, we have, for arbitrary δ $(0 < \delta < 1)$,

$$\left|\sum_{k=1}^{n}\frac{f(1)-f(x_k)}{(1-x_k)^2}\right| \leq$$

$$\leq C\frac{n}{\delta^2} + n^2 \max_{|1-x_k|\leq\delta}\left|\frac{f(1)-f(x_k)}{1-x_k}\right| .$$

with C independent of n. Thus by (4.12),

$$\left| \frac{1}{n^2} \sum_{k=1}^{n} \frac{f(1)-f(x_k)}{(1-x_k)^2} \right| \le$$

$$\le \frac{C}{n\delta^2} + \epsilon(\delta) \max_{|1-x_k|\le\delta} \left(\frac{1}{\sqrt{1-x_k}\,\log\,(1-x_k)} \right) ,$$

where $\epsilon(\delta) \to 0$ if $\delta \to 0$. If we choose $\delta = \log n/n$ then, by (3.13) $|1-x_k| \le \delta$ implies $k \le c_3 \sqrt{n \log n}$ (c_3 constant). Hence

$$\frac{1}{\sqrt{1-x_k}} = O(n) , \quad \frac{-1}{\log\,(1-x_k)} = O\,(\frac{1}{\log n}) ,$$

so that (4.2) holds and, by Theorem 4.2, $H^\circ_{2n+2}(A_1,f;x) \to f(x)$ uniformly. □

Putting together Theorem 4.1 and Corollary 4.1, we get:

Corollary 4.2. *Suppose* $f \in C[-1,1]$ *and its even and odd parts* $f_1(x)$, $f_2(x)$ *defined in* (3.6) *satisfy the conditions*:

$$(4.13) \quad \begin{cases} f_1(x) \text{ is differentiable at } x = 1 \text{ and } f'(1) = 0, \\ f_2(x) \text{ satisfies } f(1)-f(x) = o\,(\dfrac{-\sqrt{1-x}}{\log\,(1-x)}) , \quad x \to 1- . \end{cases}$$

Then $H^\circ_{2n+2}(A_1,f;x) \to f(x)$ *uniformly on* $1 \ge x \ge -1$.

5. *Convergence of* $H^\circ_{2n+1}(A_2,f;x)$.

We recall that the polynomial $H^\circ_{2n+1}(A_2,f;x)$ is defined by (2.4).

Theorem 5.1. *If* $f \in C[-1,1]$ *satisfies*

$$(5.1) \qquad \frac{1}{n^2}\sum_{k=1}^{n}\frac{f(\pm 1)-f(x_k)}{(1 \mp x_k)^2} = o\left(\frac{n}{\log n}\right)$$

then $H^{\circ}_{2n+1}(A_2,f;x) \to f(x)$ *uniformly on* $1 \geq x \geq -1$.

Here (5.1) consists of two simultaneous conditions at $x = + 1$. The proof depends on the following:

Lemma 5.1. *If* $f \in C[-1,1]$ *is even and satisfies* (5.1), *then*

$$(5.2) \qquad H^{\circ\prime}_{2n+1}(A_2,f;x_k) = o\left(\frac{n}{\log n}\right), \quad k = 0,\ldots,n+1 .$$

Proof. Since $f(x)$ is even, by symmetry of (1.1), it follows that, with the notation of §2, $c_{n+2} = 0$ so that, by (2.5) and (2.6), to prove (5.2) is the same as to prove that $e_{n+2} = o(1/\log n)$.

Using (2.1) and (2.2) since $f(x)$ is continuous and even, we get

$$F^*_{n+2} = \frac{1}{n^2}\sum_{k=1}^{n}\frac{f(x_k)}{1-x_k^2} - F_{n+2}(xf) = O(1) - F_{n+2}(xf) .$$

Now in (4.11) replacing $f(x)$ with $xf(x)$, which is an odd function, and using (2.1) and (5.1) we obtain,

$$F_{n+2}(xf) = \frac{3}{2n^2} \sum_{k=1}^{n} \frac{f(1)-x_k f(x_k)}{(1-x_k)^2}$$

$$= \frac{3}{2n^2} \sum_{k=1}^{n} \frac{f(1)-f(x_k)}{(1-x_k)^2} - \frac{3}{2n^2} \sum_{k=1}^{n} \frac{(1-x_k)f(x_k)}{(1-x_k)^2} =$$

$$= \frac{3}{2n^2} \sum_{k=1}^{n} \frac{f(1)-f(x_k)}{(1-x_k)^2} + O(1) = o(\frac{n}{\log n}) .$$

Since by (3.13) it is easy to see that $K^*_{n+2} = -\frac{n}{4} + o(n)$, it follows from (2.6) that $e_{n+2} = o(1/\log n)$. □

Proof of Theorem 5.1. As in the proof of Theorem 3.1, we see that it is enough to prove the theorem when $f(x) = f_1(x)$ and $f(x) = f_2(x)$, in which case (5.1) reduces to a single condition.

(i): $f(x) = f_1(x)$. Since $f_1(x)$ is even (see (3.6)), by Remark 4.2, Chapter II, $\mathcal{H}^\circ_{2n+1}(A_2,f_1;x)$ is even. Hence, the Hermite polynomial $H^*_{2n-1}(f_1,x)$ defined by

$$(5.3) \qquad H^*_{2n-1}(f_1,x_k) = f_1(x_k) , \quad H^{*\prime}_{2n-1}(f_1,x_k) = \mathcal{H}^{\circ\prime}_{2n+1}(A_2,f_1;x_k) , \qquad k = 1,\dots,n$$

is also even. Therefore, for some p_n,

$$(5.4) \qquad \mathcal{H}^\circ_{2n+1}(A_2,f_1,x) - H^*_{2n-1}(f_1,x) = p_n(T_n(x))^2 .$$

By the same argument used in the proof of Theorem 4.2, we see that in order to prove $\mathcal{H}^\circ_{2n+1}(A_2,f_1,x) \to f_1(x)$ uniformly, it is enough to show that (5.1) implies

$$H^{*'}_{2n-1}(f_1,x_k) = o\left(\frac{n}{\log n}\right) , \qquad k = 1,\dots,n \ .$$

This, by (5.3), follows from Lemma 5.1, and the proof for the case $f(x) = f_1(x)$ is complete.

(ii): $f(x) = f_2(x)$. Since $f_2(x)$ is odd, by Remark 4.2, Chapter II it follows that $\mathcal{H}^{\circ}_{2n+1}(A_2,f_2;x) = \mathcal{H}^{\circ}_{2n+2}(A_1,f_2;x)$. The theorem in this case reduces to Theorem 4.2. □

Using the same proof as in Corollary 4.1, we obtain:

Corollary 5.1. *If* $f \in C[-1,1]$ *satisfies the two conditions*

$$f(1) - f(x) = o\left(\frac{-\sqrt{1-x}}{\log (1-x)}\right) , \quad x \to 1- \tag{5.5}$$

$$f(-1) - f(x) = o\left(\frac{-\sqrt{1+x}}{\log (1+x)}\right) , \quad x \to -1+ \tag{5.6}$$

then $\mathcal{H}^{\circ}_{2n+1}(A_2,f;x) \to f(x)$ *uniformly on* $1 \geq x \geq -1$.

6. Comparison of $\mathcal{H}^{\circ}_{2n+3}(f,x)$, $\mathcal{H}^{\circ}_{2n+3}(A_1,f;x)$, $\mathcal{H}^{\circ}_{2n+1}(A_2,f;x)$.

The results of Theorems 3.1, 4.1 and 5.1 indicate that the operator $\mathcal{H}^{\circ}_{2n+1}(A_2,f;x)$ is more powerful than $\mathcal{H}^{\circ}_{2n+2}(A_1,f;x)$, which in turn is more powerful than $\mathcal{H}^{\circ}_{2n+3}(f,x)$. More precisely, let $\overset{\circ}{\mathcal{H}}{}^{C}_{2n+3}$ denotes the class of functions $f(x)$ (necessarily continuous on $1 \geq x \geq -1$) such that

$$H^{\circ}_{2n+3}(f,x) \to f(x), \text{ uniformly on } 1 \geq x \geq -1 ,$$

and let $\overset{\circ}{H}{}^{C}_{2n+2}(A_1)$, $\overset{\circ}{H}{}^{C}_{2n+1}(A_2)$ be similarly defined. We can formulate:

Theorem 6.1. We have

$$\text{(6.1)} \qquad \overset{\circ}{H}{}^{C}_{2n+3} \subset \overset{\circ}{H}{}^{C}_{2n+2}(A_1) \subset \overset{\circ}{H}{}^{C}_{2n+1}(A_2) .$$

Proof. (i): $\overset{\circ}{H}{}^{C}_{2n+3} \subset \overset{\circ}{H}{}^{C}_{2n+2}(A_1)$. From Theorems 3.1, 4.1 and 4.2 it follows immediately that if $f(x) = f_1(x)$ or $f(x) = f_2(x)$ (see (3.6)), then $f \in \overset{\circ}{H}{}^{C}_{2n+3}$ implies $f \in \overset{\circ}{H}{}^{C}_{2n+2}(A_1)$. Therefore, by linearity, the same is true for arbitrary $f(x)$.

(ii): $\overset{\circ}{H}{}^{C}_{2n+2}(A_1) \subset \overset{\circ}{H}{}^{C}_{2n+1}(A_2)$. By Remark 4.2, it is easy to see that whenever $f(x) = f_1(x)$ or $f(x) = f_2(x)$, $f \in \overset{\circ}{H}{}^{C}_{2n+2}(A_1)$ implies $f \in \overset{\circ}{H}{}^{C}_{2n+1}(A_2)$. Therefore, by linearity, $\overset{\circ}{H}{}^{C}_{2n+2}(A_1) \subset \overset{\circ}{H}{}^{C}_{2n+1}(A_2)$. □

Remark 6.1. Both inclusions in (6.1) are strict. More precisely, we have

(i): $\overset{\circ}{H}{}^{C}_{2n+3} \neq \overset{\circ}{H}{}^{C}_{2n+2}(A_1)$. By Remark 3.1, the function $f(x) = x$, is in $\overset{\circ}{H}{}^{C}_{2n+2}(A_1)$ by Theorem 4.2. However $f \notin \overset{\circ}{H}{}^{C}_{2n+3}$ (Berman [1], Saxena [18]).

(ii): $\overset{\circ}{H}{}^{C}_{2n+2}(A_1) \neq \overset{\circ}{H}{}^{C}_{2n+1}(A_2)$. The even function $f(x) = \sqrt{1-x^2} \,/\, \log^2(1-x)^2$ is clearly in $\overset{\circ}{H}{}^{C}_{2n+1}(A_2)$ by Theorem 5.1. However, $f(x)$ is not in $\overset{\circ}{H}{}^{C}_{2n+2}(A_1)$ by Remark 3.1. In fact, since

$f(x)$ is even, $H^{\circ}_{2n+2}(A_1,f;x) = H^{\circ}_{2n+3}(f,x)$, but $f'(x)$ is infinite at $x = 1$.

CHAPTER IV

CONVERGENCE OF SOME AVERAGING HERMITE-TYPE INTERPOLATORS ON JACOBI NODES

1. Introduction.

Let $\alpha,\beta > -1$ and let $P_n^{(\alpha,\beta)}(x)$ denote the Jacobi polynomial of degree n given by the equation

$$(1.1) \qquad (1-x^2)P_n'' + (\beta-\alpha-x(\alpha+\beta+2))P_n' + n(n+\alpha+\beta+1)P_n = 0 \ , \quad P_n \equiv P_n^{(\alpha,\beta)} \ ,$$

and normalized by the condition $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$. The zeros of $w(x) = P_n^{(\alpha,\beta)}(x)$ are denoted by:

$$(1.2) \qquad 1 > x_{1n}^{(\alpha,\beta)} > x_{2n}^{(\alpha,\beta)} > \dots > x_{nn}^{(\alpha,\beta)} > -1 \ ,$$

and for simplicity we write x_k for $x_{kn}^{(\alpha,\beta)}$.

In §2, we shall show that, if $f \in C[-1,1]$, both $H^\circ_{2n-2}(A_1,f;x)$ and $H^\circ_{2n-3}(A_2,f;x)$ converge uniformly to $f(x)$ on every closed subinterval of $1 > x > -1$. The behaviour of these two operators is the same as that of $H^\circ_{2n-1}(f,x)$, defined on the nodes (1.1). In §3, we consider the special case $\alpha = \beta$ and obtain a sufficient condition for the uniform convergence of $L_{n-3}(A_2,f;x)$ to $f(x)$. Section 4 deals with the Grünwald-type mean $L^*_{n-3}(A_2,f;x)$ of $L_{n-3}(A_2,f;x)$ and its uniform convergence to $f(x)$ on every closed subinterval of $1 > x > -1$, provided $f \in C[-1,1]$.

We shall use the following relations ([23], (8.9.1) and (8.9.2), p. 236), where the notation $u_n \sim v_n$ means $u_n = O(v_n)$ and $v_n = O(u_n)$.

$$\arccos x_k = \frac{\pi k}{n} + O\left(\frac{1}{n}\right) , \qquad k = 1,\ldots,n , \tag{1.3}$$

$$|P_n^{(\alpha,\beta)'}(x_k)| \sim \begin{cases} k^{-\alpha-3/2} \quad n^{\alpha+2} , & x_k \geq 0 \\ k^{-\beta-3/2} \quad n^{\beta+2} , & x_k < 0 . \end{cases} \tag{1.4}$$

Also, we need the quadrature formula ([23], (15.3.1), p. 349):

$$\int_{-1}^{1} g(x)(1-x)^{\alpha}(1+x)^{\beta}dx = \sum_{k=1}^{n} \lambda_k g(x_k) + R_n(g) , \tag{1.5}$$

where

$$\lambda_k = \lambda_{kn}^{(\alpha,\beta)} = \frac{t_n}{(1-x_k^2)(P_n^{(\alpha,\beta)'}(x_k))^2} , \quad k = 1,\ldots,n , \tag{1.6}$$

$$t_n = t_n^{(\alpha,\beta)} = 2^{\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)} , \tag{1.7}$$

and $R_n(g) = o(1)$ whenever $f \in C[-1,1]$.

2. Convergence of $H^{\circ}_{2n-2}(A_1,f;x)$, $H^{\circ}_{2n-3}(A_2,f;x)$.

We recall that $H^{\circ}_{2n-2}(A_1,f;x)$ and $H^{\circ}_{2n-3}(A_2,f;x)$ are the averaging Hermite interpolators with respect to $A_1(z) = 1-z$,

$A_2(z) = 1-2z+z^2$ and based on the points (1.2), whose explicit forms are given in §4, Chapter II.

Theorem 2.1. If $f \in C[-1,1]$ *and* $\alpha,\beta > -1$ *then, for all* a, b, $1 > a > b > -1$, *we have*

(2.1) $$H^\circ_{2n-2}(A_1,f;x) \to f(x) \quad \textit{uniformly on} \quad a \geq x \geq b \ ,$$

(2.2) $$H^\circ_{2n-3}(A_2,f;x) \to f(x) \quad \textit{uniformly on} \quad a \geq x \geq b \ .$$

Furthermore, if $\alpha < 0$ $(\beta < 0)$, *we can take* $a = 1$ $(b = -1)$.

The proof of (2.1) is similar to that of (2.2), hence we shall prove only (2.2). The proof depends on the following two lemmas.

Lemma 2.1. Let $w(x) = P_n^{(\alpha,\beta)}(x)$, $\alpha,\beta > -1$, *let* t_n *be given by* (1.7). *If we set* $p(x) = (1-x)^{\alpha+1}(1+x)^{\beta+1}$, *then, as* $n \to \infty$,

(2.3) $$J_n \equiv \sum_{k=1}^{n} \frac{1}{(w'(x_k))^2} = \frac{1}{t_n} \int_{-1}^{1} p(x)dx + o(1) \ ,$$

(2.4) $$J_n^1 \equiv \sum_{k=1}^{n} \frac{x_k}{(w'(x_k))^2} = \frac{1}{t_n} \int_{-1}^{1} p(x)xdx + o(1) \ ,$$

(2.5) $$K_n \equiv \sum_{k=1}^{n} \frac{k}{(w'(x_k))^2} = \frac{n}{\pi t_n} \int_{-1}^{1} p(x)(\text{arc cos}x)dx + o(n) \ ,$$

(2.6) $$K_n^1 \equiv \sum_{k=1}^{n} \frac{kx_k}{(w'(x_k))^2} = \frac{n}{\pi t_n} \int_{-1}^{1} p(x)\; x(\text{arc cos}x)dx + o(n) \;.$$

Proof. We shall not prove (2.4) and (2.6), as their derivation is analogous to that of (2.3) and (2.5) respectively.

(i). To show (2.3), let us take $g(x) = 1-x^2$ in the quadrature formula (1.5). Since $g(x)$ is continuous, (1.5) becomes

(2.7) $$\int_{-1}^{1} p(x)dx = t_n \sum_{k=1}^{n} \frac{1}{(w'(x_k))^2} + t_n\, o(1) \quad .$$

Since by Stirling's asymptotic formula $t_n \to 2^{\alpha+\beta+1} \neq 0$, on dividing (2.7) by t_n we get (2.3).

(ii). In order to prove (2.5), let us observe that by (1.3) we have $k = \frac{n}{\pi}$ arc cos $x_k + 0(1)$. Hence by (2.3)

$$\sum_{k=1}^{n} \frac{k}{(w'(x_k))^2} = \frac{n}{\pi} \sum_{k=1}^{n} \frac{\text{arc cos } x_k}{(w'(x_k))^2} + 0(1) \quad .$$

The rest follows by taking $g(x) = \text{arc cos } x$ in (1.5) and following the argument of (i). □

Lemma 2.2. *Let* $f \in C[-1,1]$ *and let* J_n, J_n^*, K_n, K_n^*, F_n, F_n^* *be given by* (4.1), *Chapter* I. *If* $\alpha,\beta > -1$, *then*:

(2.8) $$-K_nF_n^* + K_n^*F_n = 0(n) \quad ,$$

(2.9) $$J_nF_n^* - J_n^*F_n = 0(1) \quad ;$$

(2.10) $$(J_nK_n^* - J_n^*K_n)^{-1} = 0(1/n) \quad .$$

Proof. We shall divide the proof into two parts.

(i). To show (2.8) and (2.9) let us observe first that by (1.3), we have

(2.11) $$x_k = \cos\left(\frac{k\pi}{n} + 0\left(\frac{1}{n}\right)\right) = \cos\frac{k\pi}{n} + 0\left(\frac{1}{n}\right) \quad .$$

Therefore, it follows easily that

(2.12) $$s_n = \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} \cos\frac{k\pi}{n} + 0(1) = 0(1) \quad .$$

Using (4.1) (Chapter II)(2.3)-(2.6) and (2.12), we easily see that

(2.13) $$\begin{gathered} J_n = 0(1), \quad J_n^* = 2s_nJ_n - J_n^1 = 0(1) \quad , \\ K_n = 0(n), \quad K_n^* = 2s_nK_n - K_n^1 = 0(n) \quad , \end{gathered}$$

and, since $f(x)$ is continuous and $s_n = 0(1)$, $J_n = 0(1)$,

$$F_n = 0\left(\sum_{k=1}^{n}\left|\frac{w''(x_k)}{(w'(x_k))^3}\right|\right)$$

$$F_n^* = 0(1) + 0\left(\sum_{k=1}^{n}\left|\frac{w''(x_k)}{(w'(x_k))^3}\right|\right).$$

Now from (1.1) it follows that $|w''(x_k)/w'(x_k)| \le C/(1-x_k^2)$ for some C independent of k, n. Therefore, following the argument of the proof of Lemma 2.1, we see that

$$\sum_{k=1}^{n} \left| \frac{w''(x_k)}{(w'(x_k))^3} \right| \leq C \sum_{k=1}^{n} \frac{1}{(1-x_k^2)(w'(x_k))^2} =$$

$$= \frac{C}{t_n} \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta}dx + o(1) = O(1) \quad .$$

Therefore $F_n = O(1)$, $F_n^* = O(1)$ which, combined with (2.13), yields (2.8) and (2.9).

(ii). To prove (2.10), let us set

$$I = \int_{-1}^{1} p(x)x\,dx \int_{-1}^{1} p(x)(\mathrm{arc\,cos}\,x)dx -$$

$$- \int_{-1}^{1} p(x)dx \int_{-1}^{1} p(x)x(\mathrm{arc\,cos}\,x)dx \quad .$$

From (4.1) Chapter I, (2.13) and Lemma 2.1 we then obtain, since $t_n \to 2^{\alpha+\beta+1}$,

$$J_nK_n^* - J_n^*K_n = -J_nK_n^1 + J_n^1K_n = \frac{n}{\pi t_n^2} I + o(n) \quad . \tag{2.14}$$

It is clear from (2.14) that in order to prove (2.10) it is enough to show that $I \neq 0$. But, since $p(x) \geq 0$ and x, $\mathrm{arc\,cos}\,x$ are two non-constant, monotone functions, $I \neq 0$ is a consequence of the so-called Chebyshev inequality in integral form ([12], Theorem 4.3, p. 43). Lemma 2.2 is now proved. □

Proof of Theorem 2.1. By (4.4) of Chapter II, we see easily that

$$H^{\circ\prime}_{2n-1}(A_2,f;x) = O(d_n) + nO(e_n) \ , \qquad k = 1,\dots,n \ ,$$

where d_n, e_n are given by (4.5), Chapter I. Therefore from Lemma 2.2 it follows that

$$H^{\circ\prime}_{2n-1}(A_2,f;x) = O(1) \ , \qquad k = 1,\dots,n \ . \tag{2.15}$$

By ([23], Theorem 14.6, p. 338), (2.15) implies $H^{\circ}_{2n-1}(A_2,f;x) \to f(x)$ uniformly on every closed subinterval of $1 > x > -1$ (on $1 \geq x \geq -1$ if $\alpha,\beta < 0$). Therefore Theorem 2.2 is proved. ☐

3. Convergence of $L_{n-3}(A_2,f;x)$.

Here we consider the averaging interpolator $L_{n-3}(A_2,f;x)$ with respect to $A_2(z) = 1-2z+z^2$, based on the points (1.2) with $\alpha = \beta > -1$. In order to prove our main theorem, we need to recall an explicit formula for $L_{n-3}(A_2,f;x)$. Set

$$\begin{aligned} &M_n = \sum_{k=1}^{n} \frac{(-1)^{n-k}}{w'(x_k)} \ , \qquad M_n^* = \sum_{k=1}^{n} \frac{(-1)^{n-k}x_k}{w'(x_k)} \ , \\ &K_{n-1} = \sum_{k=1}^{n-1} \frac{(-1)^{n-k+1}(n-k)}{w'(x_k)} \ , \quad K_{n-1}^* = \sum_{k=1}^{n-1} \frac{(-1)^{n-k+1}(n-k)x_k}{w'(x_k)} \ , \end{aligned} \tag{3.1}$$

$$B_k \equiv B_{kn} = \frac{-K^*_{n-1} + x_k K_{n-1}}{M_n K^*_{n-1} - M^*_n K_{n-1}} \cdot \frac{1}{w'(x_k)} ,$$

(3.2)

$$C_k \equiv C_{kn} = \frac{M^*_n - x_k M_n}{M_n K^*_{n-1} - M^*_n K_{n-1}} \cdot \frac{1}{w'(x_k)} ,$$

and, with $\ell_k(x) = w(x)/(x-x_k)w'(x_k)$,

$$S_1(x) \equiv S_{1n}(x) = \sum_{k=1}^{n} (-1)^{n-k} \ell_k(x) ,$$

(3.3)

$$S_2(x) \equiv S_{2n}(x) = \sum_{k=1}^{n} (-1)^{n-k}(n-k)\ell_k(x) .$$

Then ([21], Theorem 1, p. 6)

(3.4) $$L_{n-3}(A_2,f;x) =$$

$$= \sum_{k=1}^{n} f(x_k)\{\ell_k(x)+B_k(x)S_1(x)+C_k(x)S_2(x)\} .$$

Theorem 3.1. *Let the function* $f(x)$ *satisfy the Dini-Lipschitz condition on* $1 \leq x \leq -1$, *and let* $\alpha = \beta > -1$. *Then, for every* a,b $(1 > a > b > 1)$,

(3.5) $$L_{n-3}(A_2,f;x) \to f(x) \quad \text{uniformly on} \quad a \geq x \geq b .$$

Furthermore, if $\alpha = \beta \leq 1/2$, *we can take* $a = 1$, $b = -1$.

The proof depends on the following two lemmas.

Lemma 3.1. If $\alpha = \beta > -1$ and M_n, M_n^*, K_{n-1}, K_{n-1}^*, B_{kn}, C_{kn} are given by (3.1) and (3.2), then

$$M_n \sim n^{1/2}, \quad M_n^* = 0 , \tag{3.6}$$

$$K_{n-1} \sim n^{3/2}, \quad K_{n-1}^* \sim n^{3/2} , \tag{3.7}$$

$$B_{kn} = O(1/n), \quad C_{kn} = O(1/n^2), \quad k = 1,\ldots,n . \tag{3.8}$$

Proof. We divide the proof into three parts.

(i). Since $\alpha = \beta$ and $w'(x_k) = (-1)^{k-1}|w'(x_k)|$, $k = 1,\ldots,n$, from (1.4) it follows immediately that $w'(x_k) \sim (-1)^{k-1} k^{-\alpha-3/2} n^{\alpha+2}$, $k = 1,\ldots,n$. Using this and the fact that the points (1.2) are symmetrical when $\alpha = \beta$, we easily obtain from (3.1) that $M_n^* = 0$ and

$$M_n \sim \frac{(-1)^{n-1}}{n^{\alpha+2}} \sum_{k=1}^{n} k^{\alpha+3/2} = (-1)^{n-1} \frac{n^{1/2}}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^{\alpha+3/2} .$$

Since $\alpha > -1$, we thus obtain (3.6).

(ii). From (3.1) it follows immediately, by symmetry, that

$$K_{n-1} - M_n = \sum_{k=1}^{n} \frac{(-1)^{n-k+1}(n-k+1)}{w'(x_k)} =$$

$$= \sum_{j=1}^{n} \frac{(-1)^j j}{w'(x_j)} \sim \frac{-1}{n^{\alpha+2}} \sum_{j=1}^{n} j^{\alpha+3/2} ,$$

hence, following the argument of (i), $K_{n-1} - M_n \sim n^{1/2}$. Therefore, by (3.6), $K_{n-1} \sim n^{3/2}$. Also, as $M_n^* = 0$, we see from (3.1) that

$$K^*_{n-1} - K_{n-1} = \sum_{k-1}^{n} \frac{(-1)^{n-k+1} K(1-x_k)}{w'(x_k)}$$

$$\sim \frac{(-1)^n}{n^{+2}} \sum_{k=1}^{n} (1-x_k)\, k^{\alpha+3/2} .$$

Since $0 < 1-x_k < 2$, $k = 1,\ldots,n$, the last sum is

$$O\Big(\sum_{1}^{n} k^{\alpha+3/2}/n^{\alpha+2}\Big) = O(n^{1/2}) .$$

This, together with $K_{n-1} \sim n^{3/2}$, proves (3.7).

(iii). (3.8) follows immediately from (3.2), (3.6) and (3.7). The lemma is thus proved. □

Lemma 3.2. *If* $\alpha = \beta > -1$ *we have*, *for every* a,b $(1 > a > b > -1)$:

$$\text{(3.9)} \qquad \sum_{k=1}^{n} |\ell_k(x) + B_k S_1(x) + C_k S_2(x)| \le C \log n, \quad a \ge x \ge b ,$$

where C *depends only on* a,b. *Furthermore*, *if* $\alpha = \beta \le -1/2$, *we can take* $a = 1$, $b = -1$.

Proof. From (3.3) and Lemma 3.1 it follows immediately that, for some $\underset{\sim}{C}$ independent of n, the left hand side of (3.9) is less than $C \sum_{1}^{n} |\ell_k(x)|$. Therefore Lemma 3.2 follows from a known result ([23], Theorem 14.4, §14.4).

Proof of Theorem 3.1. Let $P_{n-3}(x)$ denote the polynomial of degree n-3 of best approximation to $f(x)$ on $1 \geq x \geq -1$. If $w(f,\delta)$ is the modulus of continuity of $f(x)$, then, by the famous result of Jackson ([18], Vol. I, Theorem 1, p. 84),

$$(3.10) \qquad E_{n-3}(f) \equiv \max_{1 \leq x \leq -1} |P_{n-3}(x)-f(x)| = O(w(f, 1/n)) \quad .$$

Since $L_{n-3}(A_2,f;x)$ is linear and reproduces polynomials of degree $\leq n-3$, we obtain

$$(3.11) \qquad |L_{n-3}(A_2,f;x)-f(x)| \leq |P_{n-3}(x)-f(x)| + \\ + |L_{n-3}(A_2,f-P_{n-3};x)| \quad .$$

From (3.4) and (3.10), we see that, for $a \geq x \geq b$ $(1 \geq x \geq -1$ if $\alpha = \beta \leq -1/2)$,

$$|L_{n-3}(A_2,f;x)-f(x)| \leq C \log n \, w(f, \tfrac{1}{n}) \quad .$$

Therefore $L_{n-3}(A_2,f;x) \to f(x)$ uniformly if $f(x)$ satisfies the Dini-Lipschitz condition on $1 \geq x \geq -1$. □

4. The Grünwald-Type Means of $L_{n-3}(A_2,f;x)$.

Let an integer n and a real number α be given. Set

$$(4.1) \qquad \gamma \equiv \gamma_n(\alpha) = \frac{\pi}{2N} - \frac{\xi}{N} , \quad N = n + \alpha + \frac{1}{2} , \quad \xi = -(\alpha + \frac{1}{2}) \frac{\pi}{2} \quad .$$

For an arbitrary polynomial $Q(x)$ let us define a new polynomial

$$Q^*(x) = \frac{1}{2}\{Q(\cos[\theta+\gamma]) + Q(\cos[\theta-\gamma]), \quad x = \cos\theta \quad . \tag{4.2}$$

We shall call $Q^*(x)$, which has the same degree as $Q(x)$, the Grunwald-type mean of $Q(x)$ with respect to γ_n.

These means have been considered by Grunwald for the case when $\alpha = -1/2$ and $Q(x) = L_{n-1}(f,x)$, the Lagrange interpolator to $f(x)$ at the Chebyshev abscissas. He proved [11] that $L^*_{n-1}(f,x) \to f(x)$ uniformly on $1 \geq x \geq -1$ whenever $f \in C[-1,1]$. More recently, Vértesi [26] showed that when $L_{n-1}(f,x)$ is based on the Jacobi abscissas with $\alpha,\beta > -1$, then $L^*_{n-1}(f,x) \to f(x)$ uniformly on $a \geq x \geq b$ $(1 \geq x \geq -1$ if $\alpha,\beta \leq -1/2)$, provided $f \in C[-1,1]$. We shall prove the analogous result for the averaging interpolator $L_{n-3}(A_2,f;x)$ when $\alpha = \beta$.

Theorem 4.1. *Let* $f \in C[-1,1]$ *and let* $\alpha = \beta > -1$. *Also let* $L^*_{n-3}(A_2,f;x)$ *denote the Grünwald-type mean of* $L_{n-3}(A_2,f;x)$ *with respect to* $\gamma_n(\alpha)$. *Then for every* a,b $(1 > a > b > -1)$,

$$L^*_{n-3}(A_2,f;x) \to f(x) \quad \textit{uniformly on} \quad a \geq x \geq b \quad . \tag{4.3}$$

Furthermore, *if* $\alpha = \beta \leq -1/2$, *we can take* $a = 1$, $b = -1$.

Proof. Since $L_{n-3}(A_2,f;x)$ reproduces polynomials of degree $\leq n-3$, it is easy to see that $L^*_{n-3}(A_2,P_{n-3};x) = P^*_{n-3}(x)$. Hence, as in the proof of Theorem 3.1, we obtain

$$|L^*_{n-3}(A_2,f;x)-f(x)| \le |P^*_{n-3}(x)-f^*(x)| + |f^*(x)-f(x)|$$
$$+ |L^*_{n-3}(A_2,f-P_{n-3};x)| \le w(f,\tfrac{1}{n})(C_1+C_2+\sum_{k=1}^{n}|\ell^*_k(x)|),$$
$$1 \ge x \ge -1 \quad .$$

Using the result of [26]

$$|L^*_{n-3}(A_2,f;x) - f(x)| \le Cw(f,\tfrac{1}{n}) \ , \qquad a \ge x \ge b \ ,$$

where C depends only on a,b $(1 > a > b > -1)$. Furthermore, when $\alpha = \beta \le -1/2$ we can also take $a = 1$, $b = -1$. The proof is now complete. □

CHAPTER V

TRIGONOMETRIC AVERAGING INTERPOLATORS ON EQUIDISTANT NODES

1. Introduction.

In this chapter we consider the trigonometric averaging interpolators $T_{n-1}(A_2,f;x)$ with respect to the polynomial

$$A_2(z) = (1+rz)(1+sz) \; ; \quad r, s \text{ real} , \tag{1.1}$$

and based on the equidistant nodes

$$x_k = x_{kn} = \frac{2k\pi}{2n+1} , \qquad k = 0,\dots,2n \; . \tag{1.2}$$

After introducing some notation in §2, we state our main results in §3. We also state in §3 that the Bernstein-type mean $T^*_{n-1}(A_2,f;x)$ converges uniformly to $f(x)$ on $[0,2\pi]$ for every $f \in C[0,2\pi]$. This is the analogue of a classical result of Bernstein. In §§4 and 5 we introduce and study some auxiliary polynomials. In §6 we deal with the uniform norm of $T_{n-1}(A_2,f;x) - I_n(f,x)$, where $I_n(f,x)$ is the trigonometric interpolator to $f(x)$ at the nodes (1.2), and give the proofs of the convergence theorems of §3.

We recall that, by Theorem 5.1 of Chapter II, $T_{n-1}(A_2,f;x)$ exists and is unique for every 2π-periodic $f(x)$ if and only if

$$A_2(e^{ix_k}) \neq 0 , \qquad k = 0, \pm 1, \ldots, \pm (n-1) . \tag{1.3}$$

Since r, s are real, it is easy to see that (1.3) is equivalent to the condition $r \neq -1$ and $s \neq -1$.

2. Notations and Preliminaries.

Let $I_n(f,x)$ denote the trigonometric interpolator to $f(x)$ at the points (1.2) and let $I_{n,q}(f,x)$ denote the partial sum of order $q (0 \leq q \leq n-1)$ of $I_n(f,x)$. Also, let $a_k \equiv a_{kn}(f)$, $b_k \equiv b_{kn}(f)$, $k = 0,\ldots,n$ denote the coefficients of $I_n(f,x)$, which are given by (7.1) of Chapter II.

Since $I_n(f,x_k) = f(x_k)$, $k = 0,\ldots,2n$ and since, by definition, $T_{n-1}(A_2,f,x)$ depends only on the values $f(x_k)$, we have $T_{n-1}(A_2,f;x) = T_{n-1}(A_2,I_n;x)$. Since $I_n(f,x) = I_{n,n-1}(f,x) + a_n \cos nx + b_n \sin nx$ and since $T_{n-1}(A_2,f;x)$ reproduces polynomials of degree $\leq n-1$, we see easily that

$$\begin{aligned} T_{n-1}(A_2,f;x) = I_{n,n-1}(f,x) &+ a_n T_{n-1}(A_2, \cos nt;x) + \\ &+ b_n T_{n-1}(A_2, \sin nt;x) \quad . \end{aligned} \tag{2.1}$$

This expression simplifies further when $f(x)$ is even or odd. Since $b_{nn}(f) = 0$ if f is even and $a_{nn}(f) = 0$ if f is odd, we obtain, on writing $I_{n,n-1}(f,x) = I_n(f,x) - a_n \cos nx - b_n \sin nx$, that

$$T_{n-1}(A_2,f;x) = I_n(f,x) - a_n \cos nx + a_n T_{n-1}(A_2, \cos nt;x), \tag{2.3}$$

$$(f \text{ even}) \quad ,$$

$$T_{n-1}(A_2,f;x) = I_n(f;x) - b_n \sin nx + b_n T_{n-1}(A_2, \sin nt;x), \qquad (f \text{ odd}) . \tag{2.4}$$

Since $A_2(z)$ has real coefficients, we see immediately that, setting

$$\tau_{n-1}(x) \equiv T_{n-1}(A_2, e^{int};x) , \tag{2.5}$$

we have

$$\begin{aligned} T_{n-1}(A_2, \cos nt;x) &= \operatorname{Re}\{\tau_{n-1}(x)\} , \\ T_{n-1}(A_2, \sin nt;x) &= \operatorname{Im}\{\tau_{n-1}(x)\} . \end{aligned} \tag{2.6}$$

We shall obtain the explicit form of the polynomial $\tau_{n-1}(x)$ in §4.

In the following, we shall denote by $D_n(x)$ the Dirichlet kernel

$$D_n(x) = \frac{\sin \frac{2n+1}{2} x}{2 \sin \frac{x}{2}} = \tag{2.7}$$

$$\equiv \frac{1}{2} + \cos x + \ldots + \cos nx = \frac{1}{2} \sum_{k=-n}^{n} e^{ikx} .$$

Also, given a trigonometric polynomial $W(x)$, we introduce the trigonometric polynomial

$$W^*(x) = \frac{1}{2} \{W(x + \frac{x_1}{2}) + W(x - \frac{x_1}{2})\} , \tag{2.8}$$

which we shall call the Bernstein-type mean of $W(x)$. Such means have been considered by Bernstein who proved ([18], Vol. III, Theorem 1, p. 72) that $I_n^*(f,x) \to f(x)$ uniformly on $[0,2\pi]$, whenever $f \in C[0,2\pi]$.

3. Statement of the Main Results.

We state below our main results on the convergence of $T_{n-1}(A_2,f;x) - I_n(f,x)$, which will be proved in §7.

Theorem 3.1. *Let* $f \in C[0,2\pi]$ *be even and let* $r \neq -1$, $s \neq -1$ *be real. Then, when* $r = s = 1$, *the necessary and sufficient condition for* $T_{n-1}(A_2,f;x)$ *to tend to* $f(x)$ *uniformly on* $[0,2\pi]$ *is that* $I_n(f,x)$ *tend to* $f(x)$ *uniformly on* $[0,2\pi]$. *For* $r \neq 1$ *or* $s \neq 1$, *if*

$$a_{nn}(f) = o(1/n) \quad , \tag{3.1}$$

where $a_{nn}(f)$ *is given by* (7.1), *Chapter* II, *then*

$$T_{n-1}(A_2,f;x) - I_n(f,x) \to 0 \quad \text{uniformly on} \quad [0,2\pi] \quad . \tag{3.2}$$

Theorem 3.2. *Let* $f \in C[0,2\pi]$ *be odd, and let* $r \neq -1$, $s \neq -1$ *be real. Then the necessary and sufficient condition for* $T_{n-1}(A_2,f;x)-I_n(f,x)$ *to tend to zero uniformly on* $[0,2\pi]$ *is that:*

$$b_{nn}(f) = \begin{cases} o(1/\log n) \ , & \text{when} \quad r = 1 \quad \text{or} \quad s = 1 \ ; \\ o(1/n^2) \ , & \text{when} \quad r \neq 1 \quad \text{and} \quad s \neq 1 \ , \end{cases} \tag{3.3}$$

where $b_{nn}(f)$ *is given by* (7.1), *Chapter* II.

Theorem 3.3. *Let* $f \in C[0,2\pi]$, *and let* $T^*_{n-1}(A_2,f;x)$ *be given by* (2.8). *Then*

$$T_{n-1}(A_2,f;x) \to f(x) \quad \textit{uniformly on} \quad [0,2\pi] \;. \tag{3.4}$$

4. Explicit Form of $\tau_{n-1}(x) \equiv T_{n-1}(A_2,e^{int};x)$.

In order to obtain $\tau_{n-1}(x)$ we shall need two lemmas. Let us denote by $U_{n-1}(f,x)$ the trigonometric polynomial of degree $\leq n-1$, interpolating $f(x)$ at the points $x_0,\dots,x_{2n-2}$, that is,

$$U_{n-1}(f,x_k) = f(x_k) \;, \qquad k = 0,\dots,2n-2 \;. \tag{4.1}$$

Set $F(x) = A_2(E)f(x)$ where $Ef(x) \equiv f(x+x_1)$. Then we have:

Lemma 4.1. *Suppose* $A_2(z)$ *satisfies* (1.3) *and suppose that, for a given* $f(x)$,

$$U_{n-1}(F,x) = \sum_{k=1-n}^{n-1} u_k e^{ikx} \;, \qquad F(x) = A_2(E)f(x) \;. \tag{4.2}$$

Then

$$T_{n-1}(A_2,f;x) = \sum_{k=1-n}^{n-1} \frac{u_k e^{ikx}}{A_2(e^{ix_k})} \;. \tag{4.3}$$

This can be derived also from ([17], Theorem 4, Remark 2). However, we give a direct proof.

Proof. Let

$$V_{n-1}(x) = \sum_{k=1-n}^{n-1} u_k e^{ikx}/A_2(e^{ix_k}) .$$

Since

$$e^{ikx_j} + (r+s)e^{ikx_{j+1}} + rse^{ikx_{j+2}} = A_2(e^{ix_k}) e^{ikx_j} ,$$

it is clear that

$$V_{n-1}(x_j) + (r+s)V_{n-1}(x_{j+1}) + rs\, V_{n-1}(x_{j+2}) = U_{n-1}(F;x_j) ,$$
$$j = 0,\ldots,2n-2 .$$

Since $V_{n-1}(x)$ has degree $\le n-1$, by the uniqueness theorem of averaging interpolation (Chapter I), we have

$$V_{n-1}(x) \equiv T_{n-1}(A_2,f;x) . \qquad \square$$

$U_{n-1}(A_2(E)f,x)$ is particularly easy to determine for $f(x) = e^{inx}$. Since $A_2(E)e^{inx_k} = e^{inx_k}A_2(e^{ix_n})$ and since $e^{ix_n} = -e^{-ix_1/2}$, we have $U_{n-1}(A_2(E)e^{int},x) = A_2(-e^{-ix_1/2})U_{n-1}(e^{int},x)$. This leads to:

Lemma 4.2. Let

$$\alpha^* \equiv \alpha_n^* = \frac{-ie^{ix_1/2}}{\sin(x_1/2)} , \qquad \beta^* \equiv \beta_n^* = \frac{-ie^{ix_1}}{\sin(x_1/2)} . \tag{4.4}$$

Then

$$U_{n-1}(e^{int},x) = \alpha^* D_{n-1}(x+x_2) + \beta^* D_{n-1}(x+x_1) = \tag{4.5}$$

$$= \frac{-ie^{ix_1/2}}{\sin(x_1/2)} \sum_{k=1-n}^{n-1} \frac{e^{ix_k} + e^{ix_1/2}}{2} e^{ik(x+x_1)} .$$

Proof. The trigonometric polynomial $Q(x)$ of degree $\leq n$ given by

$$Q(x) = e^{inx} + \alpha D_n(x-x_{2n-1}) + \beta D_n(x-x_{2n}) \tag{4.6}$$

clearly interpolates e^{inx} for every α, β at the points x_k. Hence $Q(x) \equiv U_{n-1}(e^{int},x)$ if α and β are such that

$$e^{inx} + \alpha \cos n(x-x_{2n-1}) + \beta \cos n(x-x_{2n}) \equiv 0 . \tag{4.7}$$

Since $nx_{2n-1} = (2n-2)\pi + x_1$, $nx_{2n} = (2n-1)\pi + \frac{x_1}{2}$, (4.7) reduces to $e^{inx} + \alpha\cos(nx-x_1) - \beta\cos(nx - \frac{x_1}{2}) \equiv 0$, which is equivalent to the system:

$$1 + \alpha\cos x_1 - \beta\cos\frac{x_1}{2} = i + \alpha\sin x_1 - \beta\sin\frac{x_1}{2} = 0 .$$

Hence $\alpha = \alpha^*$, $\beta = \beta^*$ as given by (4.4). Since $D_n(x) = \cos nx + D_{n-1}(x)$, it is easy to see that $U_{n-1}(e^{int},x) = \alpha^* D_{n-1}(x-x_{2n-1}) + \beta^* D_{n-1}(x-x_{2n})$. This reduces to (4.5) on observing that $x_{2n-1} = 2\pi - x_2$, $x_{2n} = 2\pi - x_1$. □

Lemma 4.3. The explicit form of $\tau_{n-1}(x) \equiv T_{n-1}(A_2,e^{int};x)$ is given by

$$\tau_{n-1}(x) = \frac{-ie^{ix_1/2} A_2(-e^{-ix_1/2})}{\sin (x_1/2)} B_{n-1}(x) \quad ; \tag{4.8}$$

where

$$B_{n-1}(x) = \sum_{k=1-n}^{n-1} \frac{e^{ix_k} + e^{ix_1/2}}{2A_2(e^{ix_k})} e^{ik(x+x_1)} \tag{4.9}$$

The proof follows immediately from Lemmas 4.1 and 4.2. It is now easy to see that

$$\begin{aligned} \mathrm{Re}\{\tau_{n-1}(x)\} &= (1-rs)\mathrm{Re}\{B_{n-1}(x)\} - \mu_n \mathrm{Im}\{B_{n-1}(x)\} , \\ \mathrm{Im}\{\tau_{n-1}(x)\} &= (1-rs)\mathrm{Im}\{B_{n-1}(x)\} + \mu_n \mathrm{Re}\{B_{n-1}(x)\} , \end{aligned} \tag{4.10}$$

where

$$\mu_n = \frac{r+s - (1+rs) \cos (x_1/2)}{\sin (x_1/2)} \quad . \tag{4.11}$$

5. Uniform Norm of $\mathrm{Re}\{B_{n-1}(x)\}$, $\mathrm{Im}\{B_{n-1}(x)\}$.

For a given 2π-periodic function $f(x)$, set $||f|| = \sup_x |f(x)|$. We shall obtain some estimates for $||\mathrm{Re}\ B_{n-1}||$, $||\mathrm{Im}\ B_{n-1}||$, where $B_{n-1}(x)$ is given by (4.9).

We introduce a trigonometric polynomial $C_{n-1}(x)$ associated to $B_{n-1}(x)$ by means of the relations

$$\mathrm{Re}\{B_{n-1}(x)\} = C_{n-1}(x+x_2) + \cos \frac{x_1}{2} C_{n-1}(x+x_1) \quad , \tag{5.1}$$

(5.2) $$\operatorname{Im}\{B_{n-1}(x)\} = \sin\frac{x_1}{2}\, C_{n-1}(x+x_1)\ .$$

It is possible to show that

$$C_{n-1}(x) = \sum_{k=1-n}^{n-1} \frac{e^{ikx}}{2A_2(e^{ix_k})}\ .$$

A little calculation shows that

(5.3) $$C_{n-1}(x) = \frac{1}{2A_2(1)} + \frac{1}{2}\sum_{k=1}^{n-1} \frac{A_2(E)\cos kx}{A_2(e^{ix_k})A_2(e^{-ix_k})}\ ,$$

where $A_2(E)\cos kx = \cos kx + (r+s)\cos k(x+x_1) + rs\cos k(x+x_2)$.

Lemma 5.1. Let $r = s = 1$. Then

(5.4) $$||\operatorname{Im}B_{n-1}|| \sim n, \qquad ||\operatorname{Re} B_{n-1}|| \sim n\log n\ .$$

Proof. (i). Since $r = s = 1$, $A_2(E)\cos kx = 2(1+\cos x_k)\cos k(x+x_1)$ and $A_2(e^{ix_k})\,A_2(e^{-ix_k}) = (2\cos\frac{x_k}{2})^4$. It then follows from (5.3), after some simplification, that

(5.5) $$C_{n-1}(x) = \frac{1}{8} + \frac{1}{4}\sum_{k=1}^{n-1} \cos k\,(x+x_1)\sec^2\frac{x_k}{2}\ .$$

Since $|\cos t| \le 1$ and $\sec^2(x_k/2) > 0$, $k = 1,\ldots,n-1$, it is easy to see that

$$||C_{n-1}|| = C_{n-1}(-x_1) = \frac{1}{8} + \frac{1}{4}\sum_{k=1}^{n-1} \sec^2\frac{x_k}{2}\ .$$

Since $\frac{x_k}{2} = \frac{n-k+1/2}{2n+1}\pi - \frac{\pi}{2}$, we have $\cos\frac{x_k}{2} = \sin\frac{n-k+1/2}{2n+1}\pi \sim \frac{n-k}{n}$, $k = 1,\ldots,n-1$, hence

$$(5.6) \qquad ||C_{n-1}|| \sim \sum_{k=1}^{n-1} \left(\frac{n}{n-k}\right)^2 \sim n^2 .$$

Therefore, by (5.2), $||\mathrm{Im}\, B_{n-1}|| \sim n$.

(ii). To prove the second part of (5.4), let us first observe that, since

$$(5.7) \qquad \cos\frac{x_1}{2} = \cos\frac{\pi}{2n+1} = 1 + 0(1/n^2) ,$$

we have, by (5.6) and (5.1),

$$\mathrm{Re}\{B_{n-1}(x)\} = C_{n-1}(x+x_2) + C_{n-1}(x+x_1) + 0(1) .$$

Since $\cos k(x+x_3) + \cos k(x+x_2) = 2\cos k(x + \frac{x_5}{2})\cos\frac{x_k}{2}$, we see from (5.5) that

$$C_{n-1}(x+x_1) + C_{n-1}(x) = \frac{1}{4} + \frac{1}{2}\sum_{k=1}^{n-1} \cos k\,(x - \frac{x_1}{2})\sec\frac{x_k}{2} .$$

Since $\sec\frac{x_k}{2} > 0$, $k = 1,\ldots,n-1$ then, as above,

$$||C_{n-1}(x+x_1) + C_{n-1}(x)|| = \frac{1}{4} + \frac{1}{2}\sum_{k=1}^{n-1} \sec\frac{x_k}{2}$$

$$\sim \sum_{k=1}^{n-1} \frac{n}{n-k} \sim n \log n .$$

Therefore, by (5.1),

$$\|\mathrm{Re}\, B_{n-1}\| \sim n \log n \;. \qquad \square$$

Lemma 5.2. If $r = s = 1$ and if

$$B^*_{n-1}(x) = B_{n-1}\left(x + \frac{x_1}{2}\right) + B_{n-1}\left(x - \frac{x_1}{2}\right) , \tag{5.8}$$

then

$$\|\mathrm{Im}\, B^*_{n-1}\| \sim \log n, \qquad \|\mathrm{Re}\, B^*_{n-1}\| \sim n \;. \tag{5.9}$$

Proof. (i). From (5.3), since $\cos k\left(x + \frac{x_1}{2}\right) + \cos k\left(x - \frac{x_1}{2}\right) = 2 \cos kx \cos \frac{x_k}{2}$, it follows that

$$C^*_{n-1}(x) \equiv C_{n-1}\left(x + \frac{x_1}{2}\right) + C_{n-1}\left(x - \frac{x_1}{2}\right) = \tag{5.10}$$

$$= \frac{1}{4} + \frac{1}{2} \sum_{k=1}^{n-1} \cos k\left(x + \frac{x_1}{2}\right) \sec \frac{x_k}{2} \;.$$

Hence, as in the proof above, part (ii), $\|C^*_{n-1}\| \sim n \log n$. Therefore, by (5.2), $\|\mathrm{Im}\, B^*_{n-1}\| \sim \log n$.

(ii). Since $\mathrm{Re}\, B^*_{n-1}(x) = C^*_{n-1}(x+x_2) + \cos \frac{x_1}{2} C^*_{n-1}(x+x_1)$ by (5.1), (5.8) and (5.10), we easily obtain, on using (5.7) and $\|C^*_{n-1}\| \sim n \log n$, that

$$\mathrm{Re}\{B^*_{n-1}(x)\} = C^*_{n-1}(x+x_2) + C^*_{n-1}(x+x_1) + 0(\log n/n) \;.$$

The rest follows as above. $\square$

Lemma 5.3. *Let* $r = 1$ *and let* s *be real*, $s \neq 1$ and $s \neq -1$. *Then*

$$\|\mathrm{Im}\, B_{n-1}\| \sim \log n, \qquad \|\mathrm{Re}\, B_{n-1}\| \sim n . \tag{5.11}$$

Proof. We shall assume $s > 1$, the proof when $s < 1$ being similar.

(i). Since $A_2(E) \cos kx = 4 \cos k(x+x_1) \cos^2 \frac{x_k}{2} + 2(s-1) \cos k\left(x + \frac{x_3}{2}\right) \cos \frac{x_k}{2}$ and $A_2(e^{ix_k})\, A_2(e^{-ix_k}) = 4 \cos^2 \frac{x_k}{2} \left(1+2s \cos \frac{x_k}{2} + s^2\right)$, we have

$$C_{n-1}(x) = \frac{1}{4(1+s)} + \sum_{k=1}^{n-1} \frac{\cos k(x+x_1)}{g(x_k)} + \frac{s-1}{2} \sum_{k=1}^{n-1} \frac{\cos k(x+x_3/2)}{g(x_k) \cos (x_k/2)} , \tag{5.12}$$

where $g(x) = 1 + 2s \cos x + s^2$. Since $s > 1$, it is easy to see that there are two constants $K_1 > K_2 > 0$ such that $K_1 > g(x) > K_2$ for all x. Therefore, as in the proof of Lemma 5.1,

$$\left\| \sum_{k=1}^{n-1} \frac{\cos kx}{g(x_k)} \right\| \sim n, \qquad \left\| \sum_{k=1}^{n-1} \frac{\cos kx}{g(x_k) \cos (x_k/2)} \right\| \sim n \log n .$$

By (5.12), this implies $\|C_{n-1}\| \sim n \log n$. Therefore, by (5.2), $\|\mathrm{Im}\, B_{n-1}\| \sim \log n$.

(ii). From (5.7), since $\|C_{n-1}\| \sim n \log n$, it follows that

$$\mathrm{Re}\{B_{n-1}(x)\} = C_{n-1}(x+x_2) + C_{n-1}(x+x_1) + O(\log n/n) . \tag{5.13}$$

From (5.12) we see easily that

$$C_{n-1}(x+x_2) + C_{n-1}(x+x_1) =$$

$$= \frac{1}{2(1+s)} + 2 \sum_{k=1}^{n-1} \frac{\cos k(x+x_5/2) \cos (x_k/2)}{g(x_k)} +$$

$$+ (s-1) \sum_{k=1}^{n-1} \frac{\cos k(x+x_7/2)}{g(x_k)} .$$

Since $\cos \frac{x_k}{2} \sim \frac{n-k}{n}$, we have $||C_{n-1}(x+x_2) + C_{n-1}(x+x_1)|| \sim n$. Hence, by (5.13),

$$||\text{Re } B_{n-1}|| \sim \log n \quad . \qquad \Box$$

In order to prove the next lemma, let us observe that, since $A_2(E)\, e^{ikx} = A_2(e^{ix_k})\, e^{ikx}$, it follows from (5.13) that

$$A_2(E)\, C_{n-1}(x) = \frac{1}{2} \sum_{k=1}^{n-1} e^{ikx} = D_{n-1}(x) \quad . \tag{5.14}$$

Lemma 5.4. Let $r \neq \pm 1$ and $s \neq \pm 1$ be real numbers. Then

$$||\text{Im } B_{n-1}|| \sim 1, \qquad ||\text{Re } B_{n-1}|| \sim n \ . \tag{5.15}$$

Proof. We prove only the first estimation of (5.15), since the second part is proved similarly. We shall show that $||C_{n-1}|| \sim n$, which by (5.2) is equivalent to $||\text{Im } B_{n-1}|| \sim 1$.

From (5.14) we obtain, by the triangle inequality

$$n - \frac{1}{2} = ||D_{n-1}|| \leq (1+|r|)(1+|s|)||C_{n-1}|| \quad .$$

In order to prove the converse inequality, let us set $P_{n-1}(x) = (1+rE)C_{n-1}(x)$. From (5.14) it follows that

$$P_{n-1}(x) + sP_{n-1}(x+x_1) = D_{n-1}(x) \quad .$$

Hence by the triangle inequality, we see that

$$n - \frac{1}{2} \geq |D_{n-1}(x)| \geq |P_{n-1}(x)| - |s|\ |P_{n-1}(x)|, \quad \text{for all } x. \tag{5.16}$$

Since $s \neq \pm 1$, it follows easily from (5.16) that $n - \frac{1}{2} \geq (1-|s|)||P_{n-1}||$. Since $C_{n-1}(x) + rC_{n-1}(x+x_1) = P_{n-1}(x)$ we obtain, by a similar argument, $||P_{n-1}|| \geq (1-|r|)||C_{n-1}||$. Hence

$$n - \frac{1}{2} \geq (1-|r|)(1-|s|)||C_{n-1}||$$

and the lemma is proved. □

6. Proofs of Theorems 3.1, 3.2 and 3.3.

Proof of Theorem 3.1. When $f \in C[-1,1]$ it is known that $a_{nn}(f) = o(1)$. From (2.3) it is easy to see that in order to prove the first part of Theorem 3.1, it is enough to show that

$$||T_{n-1}(A_2,f;x) - I_n(f,x)|| = O(1)\ a_{nn}(f) \quad . \tag{6.1}$$

From (5.10) and (4.11) we see that since, when $r = s = 1$

$$(6.2) \qquad \mu_n = 2\left(1 - \cos\frac{x_1}{2}\right) \csc\frac{x_1}{2} = \frac{x_1}{4} + o\left(\frac{1}{n}\right) = O\left(\frac{1}{n}\right) ,$$

$$\mathrm{Re}\{\tau_{n-1}(x)\} = -\mu_n \, \mathrm{Im}\{B_{n-1}(x)\} = O\left(\frac{1}{n}\right) \mathrm{Im}\{B_{n-1}(x)\}.$$

Therefore, by Lemma 5.1,

$$\mathrm{Re}\{\tau_{n-1}(x)\} = T_{n-1}(A_2, \cos nt; x) = O(1) .$$

(6.1) now follows from (2.3).

In order to prove the second part of Theorem 3.1, we shall show that

$$(6.3) \qquad ||T_{n-1}(A_2,f;x) - I_n(f,x)|| = O(n)\, a_{nn}(f) .$$

We divide the proof into two parts.

(i): $r = 1,\ s \neq 1$. From (4.11) we see that $\mu_n = (1+s)\left(1 - \cos\frac{x_1}{2}\right) \csc\frac{x_1}{2} = O\left(\frac{1}{n}\right)$, hence

$$\mathrm{Re}\{\tau_{n-1}(x)\} = (1-s)\, \mathrm{Re}\{B_{n-1}(x)\} + O\left(\frac{1}{n}\right) \mathrm{Im}\{B_{n-1}(x)\} .$$

Therefore, by Lemma 5.3, $||\mathrm{Re}\ \tau_{n-1}|| \sim n$ which, by (2.6) proves (6.3). The case $r \neq 1,\ s = 1$ is similar.

(ii): $r \neq 1,\ s \neq 1$. We see from (4.10) and (4.11) that $\mu_n = O(n)$. Therefore, by Lemma 5.4 and (4.10), $||\mathrm{Re}\ \tau_{n-1}|| = O(n)$, which implies (6.3). □

Proof of Theorem 3.2. We first consider the case when at least one among r and s is one. In order to prove this, we shall show that

$$(6.4) \qquad ||T_{n-1}(A_2,f;x) - I_n(f,x)|| \sim \log n \; b_{nn}(f) \quad .$$

We divide the proof of (6.4) into two parts.

(i): $r = s = 1$. From (6.2), since $x_1 = 2\pi/(2n+1)$, we see that $\mu_n \sim 1/n$, hence by (4.10),

$$\mathrm{Im}\{\tau_{n-1}(x)\} = T_{n-1}(A_2,\sin nt;x) \sim \frac{1}{n}\,\mathrm{Re}\{B_{n-1}(x)\} \; .$$

Then (6.4) follows from Lemma 5.1 and (2.4).

(ii): $r = 1$, $s \neq 1$. The proof follows the same lines as in Lemma 6.2, part (ii), with $\mathrm{Re}\{\tau_{n-1}(x)\}$ replaced by $\mathrm{Im}\{\tau_{n-1}(x)\}$.

We now consider the case when $r \neq 1 \neq s$. In order to prove the theorem in this case, we shall show that

$$(6.5) \qquad ||T_{n-1}(A_2,f;x) - I_n(f,x)|| \sim n^2 \, b_{nn}(f) \quad .$$

Since $\cos\frac{x_1}{2} = 1 + O(1/n^2)$ and $r \neq 1 \neq s$, it is easy to see that $r+s - (1+rs)\cos\frac{x_1}{2} \sim 1$. Hence $\mu_n \sim 1/n$ which, by Lemma 5.3 and (4.10) shows that $||\mathrm{Im}\,\tau_{n-1}|| \sim n^2$. Thus (6.5) is proved. □

Proof of Theorem 3.3. Since $a_{nn}(f) = o(1)$, $b_{nn}(f) = o(1)$ when $f(x)$ is continuous, it will be enough to show that

$$(6.6) \qquad ||T^*_{n-1}(A_2,f;x) - I^*_n(f,x)|| = O(|a_{nn}(f)|+|b_{nn}(f)|) \quad ,$$

where $T^*_{n-1}(A_2,f;x)$ and $I^*_n(f,x)$ are defined as in (2.8). By (2.1) it is clearly enough to show that

$$(6.7) \qquad ||\mathrm{Re}\ \tau^*_{n-1}|| = O(1) \ , \qquad ||\mathrm{Im}\ \tau^*_{n-1}|| = O(1) \ .$$

When $r = s = 1$, we see from (4.5) and (4.6) that

$$\mathrm{Re}\{\tau^*_{n-1}(x)\} = -\mu_n \ \mathrm{Im}\{B^*_{n-1}(x)\} = O(\tfrac{1}{n}) \ \mathrm{Im}\{B^*_{n-1}(x)\} \ ,$$

$$\mathrm{Im}\{\tau^*_{n-1}(x)\} = \mu_n \ \mathrm{Re}\{B^*_{n-1}(x)\} \ ,$$

hence (6.7) follows from Lemma 5.2. □

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